EECS 16A Designing Information Devices and Systems I
Spring 2018 Homework 4

This homework is due February 19, 2018, at 23:59.
Self-grades are due February 22, 2018, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. Finding Null Spaces

(a) Consider the column vectors of any 3×5 matrix. What is the maximum possible number of linearly independent vectors you can pick from these column vectors?

Solution:

Since the column vectors are in $\mathbb{R}^3$, there are at most 3 linearly independent vectors. Hence we can say that the column space of this matrix has dimension of at most 3.

(b) Suppose we have the following 3×5 matrix after row reduction:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

What is the minimum number of vectors spanning the column space of $\mathbf{A}$? Find a set of such vectors.

Solution:

For any vector $\vec{x}$, $\mathbf{A}\vec{x}$ is a linear combination of the columns of $\mathbf{A}$, thus the column space of $\mathbf{A}$ is a linear combination of its columns. We can see that there are only two linearly independent columns because the third component for each column vector is 0. Therefore a set of linearly independent vectors spanning the range of $\mathbf{A}$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

(c) Recall that for every vector $\vec{x}$ in the null space of $\mathbf{A}$, $\mathbf{A}\vec{x} = \vec{0}$. The dimension of a the null space is the minimum number of vectors needed to span it. Find vectors that span the null space of $\mathbf{A}$ (the matrix in the previous part). What is the dimension of the null space of $\mathbf{A}$?

Solution:

Finding the null space of $\mathbf{A}$ is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$x_1 + x_2 - 2x_4 + 3x_5 = 0$$
$$x_3 - x_4 + x_5 = 0$$

We have 5 unknowns but only 2 linearly independent equations. Therefore, there are 3 degrees of freedom in the null space. Hence, the dimension of the null space is 3. Note that because of the way Gaussian elimination is performed, x_1 and x_3 only appear once in each equation at the head of their respective rows/equations. Thus, we let x_2, x_4 and x_5 be free variables a, b and c. Now we rewrite the equations as:

$$\begin{aligned} x_1 &= -a + 2b - 3c \\ x_2 &= a \\ x_3 &= b - c \\ x_4 &= b \\ x_5 &= c \end{aligned}$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of $\mathbf{A}$ is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Look more closely at these vectors. Notice anything about their entries as compared to the row-reduced matrix? (Remember, here the row reduction went both down and then back up so that the columns that actively participated in row reduction have only one 1 in them and that 1 leads its row.) They have 1s in certain positions that represent the free variables for their respective vectors and the other vectors have 0s in those positions. Meanwhile, all the other entries have their signs flipped from the corresponding little column in the fully row-reduced matrix. This is not a coincidence and if you look closely at the pattern of the derivation above, you will see why this must always be the case.

(d) Find vector(s) that span the null space of the following matrix:

$$\mathbf{B} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution:

Using Gaussian elimination, we can row-reduce the matrix:

$$\begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vectors in the null space satisfy the following equations:

$$\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$$x_1 - 2x_2 = 0$$
$$x_3 + 2x_4 = 0$$

We then set x_2 and x_4 to be free variables and substitute in:

$$x_1 = 2a$$
$$x_2 = a$$
$$x_3 = -2b$$
$$x_4 = b$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Therefore, the null space of the matrix is spanned by the vectors:

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Notice the same pattern as before in terms of the relationship of the null space basis found to the fully row-reduced matrix.

2. Traffic Flows

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flows t_1 as the rate of cars (cars/hour) on the road between B and A, t_2 as the rate on the road between C and B and t_3 as the rate on the road between C and A.

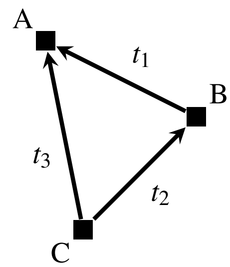


Figure 1: A simple road network.

(Note: The directions of the arrows in the figure are only the way that we define the flow by convention. If there were 100 cars per hour traveling from A to C, then $t_3 = -100$.)

We assume the "flow conservation" constraints: the total number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3).

Solution:

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

(b) Now suppose we have a larger network, as shown in Figure 2.

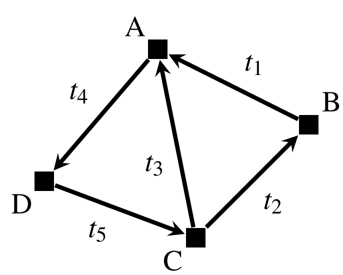


Figure 2: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors.

A Berkeley student claims that we need two sensors placed on the roads AD and BA. A Stanford student claims that we need two sensors placed on the roads CB and BA. Is it possible to determine all traffic flows with the Berkeley student's suggestion? How about the Stanford student's suggestion?

Solution:

The Stanford student is wrong (obviously). Observing t_1 and t_2 is not sufficient, as t_3, t_4 and t_5 can still not be uniquely determined. Specifically, for any $t \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$t_4 = t$$
$$t_5 = t$$
$$t_3 = t - t_1$$

On the other hand, if we're given t_1 and t_4, we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. We know that t_2 is the same as t_1 and that t_4 is the same as t_5 since the flow going into B and D must equal the flow going out. The flow into A, $t_1 + t_3$, must equal the flow going out, t_4, so:

$$t_2 = t_1$$
$$t_5 = t_4$$
$$t_3 = t_4 - t_1$$

This is related to the fact that t_1 and t_4 are parts of different cycles in the graph, whereas t_1 and t_2 are in the same cycle, so measuring both of them would not give additional information.

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.
Find a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$

$$\begin{bmatrix} & & \\ & \mathbf{B} & \\ & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network in Figure 2.
Hint: Each row is the constraint of an intersection. You can construct $\mathbf{B}$ using only $0, 1$, and -1 entries. This matrix is called the **incidence matrix.** What constraint does each column of $\mathbf{B}$ represent?
Solution:

$$\mathbf{B} = \begin{bmatrix} -1 & 0 & -1 & +1 & 0 \\ +1 & -1 & 0 & 0 & 0 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & -1 & +1 \end{bmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$
$$\qquad t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5$$

(The rows of this matrix can be in any order and your solution can differ by a factor of -1) Each row represents an intersection, and each column represents a road between two intersections. Each -1 on a row represents a road flowing into an intersection, and each $+1$ represents a road flowing out of an intersection. Each $+1$ in a column represents the source intersection of a road, and each -1 in a column represents the destination intersection of a road.

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of traffic flows for the network of Figure 2. Specifically, express this space as the span of two linearly independent vectors.

Hint: Use the claim of the correct student in part (b).

Solution:

Suppose we have a set of valid flows $\vec{t}$. Then, for any intersection, the net flow into it is the same as the net flow out of it. If we scale $\vec{t}$ by a constant a, each t_i will also get scaled by a. The net flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows $\vec{t}_1$ and $\vec{t}_2$ to get $\vec{t} = \vec{t}_1 + \vec{t}_2$. For any intersection I,

$$\text{net flow into } I = \text{net flow into } I \text{ from } \vec{t}_1 + \text{net flow into } I \text{ from } \vec{t}_2$$
$$\text{net flow out of } I = \text{net flow out of } I \text{ from } \vec{t}_1 + \text{net flow out of } I \text{ from } \vec{t}_2$$

Since the net flow into I from $\vec{t}_1$ is the same as the net flow out of I from $\vec{t}_1$ and similarly for $\vec{t}_2$, the net flow into I is the same as the net flow out of I. Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

To determine the subspace of traffic flows for the above network, use the solution in the previous part to see what $\vec{t}$ looks like in terms of $t_1 = \alpha$ and $t_4 = \beta$:

$$\vec{t} = \begin{bmatrix} \alpha \\ \alpha \\ \beta - \alpha \\ \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \vec{u}_1 + \beta \vec{u}_2$$

Clearly, $\vec{u}_1$ and $\vec{u}_2$ are linearly independent, and the space of all possible traffic flows is the span of them.

(e) Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. Use Gaussian elimination to determine the dimension of the null space of $\mathbf{B}$ and compute a basis for the null space. Does this match your answer to part (d)? Can you interpret the dimension of the null space of $\mathbf{B}$ for the road networks of Figure 1 and Figure 2?

Solution:

After row-reducing, we get the following matrix:

$$\begin{bmatrix} +1 & 0 & +1 & 0 & -1 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the column rank is 3, the dimension of the null space is 2. We can find the following basis (we use the observation that we had made that connects the basis vectors to the row-reduced matrix: t_3 and

t_5 are the free variables and so get 1s with the rest coming from sign flipping.) for the null space:

$$a\begin{bmatrix}-1\\-1\\1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix}$$

This does not match the answer in the earlier part because these are two different bases, but the null space they span is the same.

By itself, the first vector weighted by a is clearly a vector corresponding to the small cycle in the graph. The second one b is corresponds to the bigger cycle. These two cycles are still independent of each other, which is why the dimension of the null space can be interpreted as the number of "independent cycles" in the graph.

It is fine to give yourself full credit as long as you found a basis for the null space. It doesn't have to be this particular one.

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along *any* k roads is always sufficient to recover the exact flows? Prove or give a counterexample.

Hint: Consider the Stanford student.

Solution:

No, consider the network of Figure 2. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring t_1 and t_2 (as the Stanford student suggested) is not sufficient, as you showed in part (b).

(g) Let G be a network of n roads with the incidence matrix $\mathbf{B}_G$, which has a k-dimensional null space. We would like to characterize exactly when measuring the flows along a set of k roads is sufficient to recover the exact flow along all roads. To do this, it will help to generalize the problem and consider measuring *linear combinations* of flows. Let t_i be the flow on one road. We measure some linear combination of t_i's or $m_0 \cdot t_0 + m_1 \cdot t_1 + \cdots + m_n \cdot t_n$. Now we measure many of these linear combinations, which we will represent using matrix vector multiplication. Then, making k measurements is equivalent to observing the vector $\mathbf{M}\vec{t}$ for some $k \times n$ "measurement matrix" $\mathbf{M}$.

For example, for the network of Figure 2, the measurement matrix corresponding to measuring t_1 and t_4 (as the Berkeley student suggests) is:

$$\mathbf{M} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\end{bmatrix}$$

Similarly, the measurement matrix corresponding to measuring t_1 and t_2 (as the Stanford student suggests) is:

$$\mathbf{M} = \begin{bmatrix}1 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0\end{bmatrix}$$

For general networks G and measurements M, give a condition for when the exact traffic flows can be recovered in terms of the null space of $\mathbf{M}$ and the null space of $\mathbf{B}_G$.

Hint: Recovery will fail iff there are two valid flows with the same measurements, that is, there exist distinct $\vec{t}_1$ and $\vec{t}_2$, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$. Can you express this in terms of the null spaces of $\mathbf{M}$ and $\mathbf{B}_G$?

Solution:

As stated in the hint, we cannot uniquely determine the flow iff there are two valid flows that yield the same set of measurements. That is, there should not be two *distinct* valid flows $\vec{t}_1$ and $\vec{t}_2$, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$, or equivalently, such that $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = 0$.

The set of valid flows is the null space of $\mathbf{B}_G$, denoted $\text{Null}(\mathbf{B}_G)$. So recovery fails if $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = \vec{0}$ for some $\vec{t}_1, \vec{t}_2 \in \text{Null}(\mathbf{B}_G)$, with $\vec{t}_1 \neq \vec{t}_2$. The set of valid flows is a subspace, so we can equivalently state this as: Recovery fails iff $\mathbf{M}\vec{t} = \vec{0}$ for some $\vec{t} \neq \vec{0}$, $\vec{t} \in \text{Null}(\mathbf{B}_G)$.

In other words, *there should be no vector $\vec{t} \neq 0$ that is both in the null space of $\boldsymbol{B}_G$* and *in the null space of $\boldsymbol{M}$*.

This can also be stated as: *We can recover the exact traffic flows iff the null space of $\boldsymbol{B}_G$ does not non-trivially intersect the null space of $\boldsymbol{M}$*.

Full credit for stating any condition that is equivalent to this, using the null spaces of $\mathbf{M}$ and $\mathbf{B}_G$.

(h) Express the condition of the previous part in a way that can be checked computationally. For example, suppose we are given a huge road network G of all roads in Berkeley, and we want to find if our measurements M are sufficient to recover the flows.

Hint: Consider a matrix $\mathbf{U}$ whose columns form a basis of the null space of $\mathbf{B}_G$. Then $\{\mathbf{U}\vec{x} \mid \vec{x} \in \mathbb{R}^k\}$ is exactly the set of all possible traffic flows, that is, every valid flow $\vec{t}$ can be represented as $\vec{t} = \mathbf{U}\vec{x}$ for some $\vec{x}$. How can we represent measurements on these flows?

Solution:

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$. Then, as in the hint, the set $\{\mathbf{U}\vec{x} \mid \vec{x} \in \mathbb{R}^k\}$ is exactly the set of all possible traffic flows.

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the result of measuring this flow is $\mathbf{M}\vec{t} = \mathbf{MU}\vec{x}$. Now, recovering the exact flow from our measurements is equivalent to recovering $\vec{x}$ from $\mathbf{MU}\vec{x}$. Notice that the matrix $\mathbf{MU}$ is $k \times k$, so we can recover the exact flows iff $\mathbf{MU}$ is invertible. This condition can be easily checked computationally (for example, by row-reducing $\mathbf{MU}$).

Remark: Notice how defining the matrix $\mathbf{U}$ allowed us to work with flows in terms of their low-dimensional representations ($\vec{x}$), instead of dealing directly with all their components.

(i) If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can always pick a set of k roads such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

Solution:

Yes.

Let $\mathbf{U}$ be a matrix whose columns form a basis of the null space of $\mathbf{B}_G$, as above. The k columns of $\mathbf{U}$ are linearly independent since they form a basis. Since there are k linearly independent columns, when we run Gaussian elimination on $\mathbf{U}$, we must get k pivots. (Recall that "pivot" is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of $\mathbf{U}$ is k dimensional since there are some k linearly independent rows in $\mathbf{U}$ — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{t} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some k linearly independent rows of $\mathbf{U}$. By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find $\vec{x}$ from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

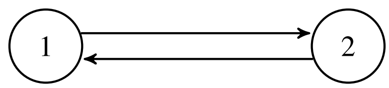
This isn't the only set of k roads that will work. But it does provide a set of k roads that are guaranteed to work.

3. Counting The Paths of a Random Surfer

In class, we discussed the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from a page to another page. To do this, we represent the webpages as a graph. If page 1 has a link to page 2, we have a directed edge from page 1 to page 2. This graph can further be represented by what is known as an "adjacency matrix", $\mathbf{A}$, with elements a_{ij}. We define $a_{ji} = 1$ if there is link from page i to page j. Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j.

This path counting aspect actually is an implicit part of the how the "importance scores" for each webpage are described. Recall that the "importance score" of a website is the steady-state frequency of the fraction of people on that website.

Consider the following graphs.



Graph A

(a) Write out the adjacency matrix for graph A.

Solution:

The adjacency matrix only has 0s and 1s, representing whether a connection between two nodes exists or not. Therefore, it can be thought of as a way to represent the connectivity of the graph. Let $\mathbf{A}$ be the adjacency matrix for graph A. Then,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(b) For graph A: How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

Solution:

We take the n^{th} power of the adjacency matrix to determine how many n-hops paths exist between the pages.

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, there is 1 one-hop path between webpage 1 and webpage 2 (which can be checked trivially).

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There are no two-hop paths between webpage 1 and webpage 2! This matches the structure of the graph since two hops will always get the websurfer back to the page they started from.

Why does this work? Let's look at what the $\mathbf{A}^2$ matrix multiplication does:

The first element (describing the path to get to and from node 1 in two hops) is (number of paths from node 1 to node 1)^2+(number of paths from node 2 to node 1)(number of paths from node 1 to node 2).

This is $0+(1)(1)=1$. The result is therefore the sum of any self-loops and the number of paths going to node 2 and back. A similar formula applies for the n^{th} powers.

This is because the concatenation of two paths is a valid path if one ends where the other begins. So the number of n-hop paths from i to j must in fact be the sum, over all intermediate pages k, of the number of ℓ-hop paths from i to k times the number of $(n-\ell)$-hop paths from k to j. This is precisely what matrix multiplication does.

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There is 1 three-hop path between webpage 1 and webpage 2. Note that $\mathbf{A}^3 = \mathbf{A}$.

(c) For graph A: What are the importance scores of the two webpages?

Solution:

To determine the importance score of the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state. Therefore, we use a transition matrix that deals with probabilities, instead of the connectivity (adjacency) matrix.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To determine the eigenvalues of this matrix:

$$\det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = 0$$

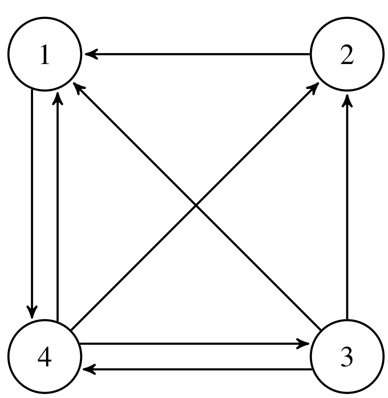
$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

The sum of the values of the vector should equal 1, so our conditions are:

$$v_1 + v_2 = 1$$
$$v_1 = v_2$$

The importance score vector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has an importance score of 0.5.



Graph B

(d) Write out the adjacency matrix for graph B.

Solution:

Let $\mathbf{B}$ be the adjacency matrix for graph B. Then,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

(e) For graph B: How many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2?

Solution:

Using the same procedure as part (b),

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

There is 1 two-hop path from webpage 1 to webpage 3. Let's look at what the $\mathbf{B}^2$ matrix multiplication does:

The B_{31} element (describing the path to get from webpage 1 to webpage 3) is

(paths from node 1 to node 3)(paths from node 1 to node 1)+(paths from node 2 to node 3)(paths from node 1 to node 2)+(paths from node 3 to node 3)(paths from node 1 to node 3)+(paths from node 4 to node 3)(paths from node 1 to node 4)+(paths from node 5 to node 3)(paths from node 1 to node 5).

This is $0+0+0+1=1$. A similar formula applies for the n^{th} powers.

For three hop paths,

$$\mathbf{B}^3 = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{bmatrix}$$

There is 1 three-hop path from webpage 1 to webpage 2.

(f) For graph B: What are the importance scores of the webpages? You may use your IPython notebook for this.

Solution:

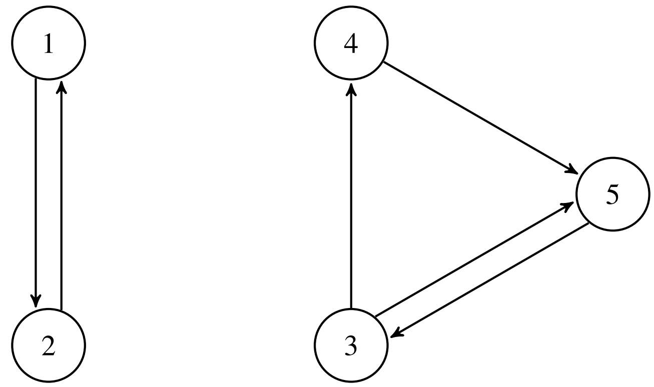
To determine the importance scores, we need to create the transition matrix $\mathbf{T}$ first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

The eigenvector associated with eigenvalue 1 is $\begin{bmatrix} -0.61 & -0.31 & -0.23 & -0.69 \end{bmatrix}^T$ (found using IPython).

Scaling it appropriately so the elements add to 1, we get $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{8} & \frac{3}{8} \end{bmatrix}^T$

These are the importance scores for the pages.



Graph C

(g) Write out the adjacency matrix for graph C.

Solution:

Let $\mathbf{C}$ be the adjacency matrix for graph C. Then,

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

(h) For graph C: How many paths are there from webpage 1 to webpage 3?

Solution:

There are no paths from webpage 1 to webpage 3 (and no n-hop paths either).

(i) For graph C: What are the importance scores of the webpages? How is graph (c) different from graph (b), and how does this relate to the importance scores and eigenvalues and eigenvectors you found?

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

The eigenvalues of this graph are $\lambda = 1, 1, -1, -\frac{1}{2} + -\frac{i}{2}, -\frac{1}{2} - -\frac{i}{2}$ (found using IPython). The eigenvectors associated with $\lambda = 1$ are $\begin{bmatrix} 0 & 0 & 0.4 & 0.2 & 0.4 \end{bmatrix}^T$ and $\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \end{bmatrix}^T$.

Why are there two eigenvectors? The first eigenvector describes the importance scores of the last three webpages, and the second vector describes the importance scores of the first two webpages. This makes sense since there are essentially “two internets", or two disjoint set of webpages. Surfers cannot transition between the two, so you cannot assign importance scores to webpage 1 and webpage 2 relative to the rest.

Assuming that each set of importance scores needs to add to 1, the first assigns importance scores of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns importance scores of 0.5 to both webpage 1 and webpage 2.

4. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.