This homework is due March 19, 2018, at 23:59.
Self-grades are due March 22, 2018, at 23:59.

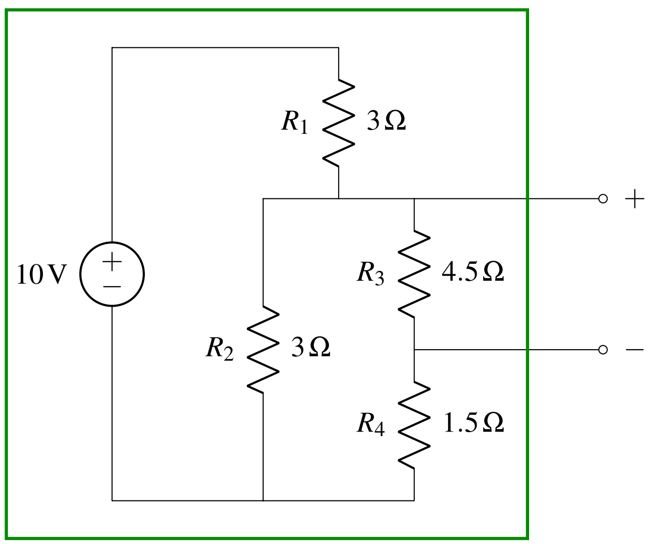
Submission Format
Your homework submission should consist of **one** file.

- `hw8.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

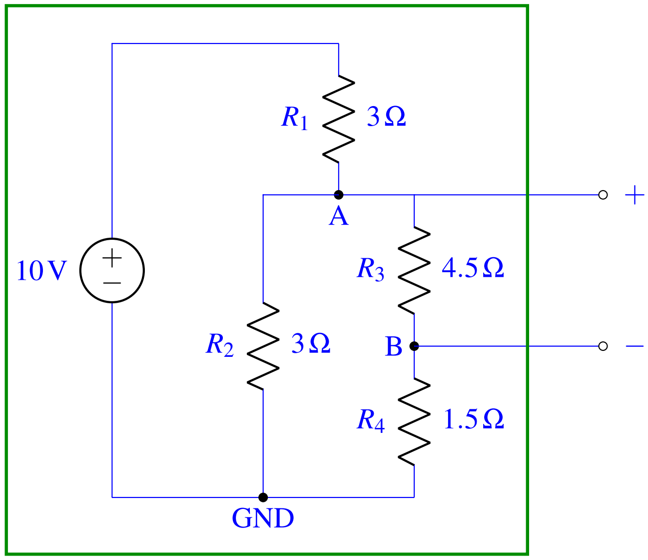
Submit the file to the appropriate assignment on Gradescope.

1. (PRACTICE) Thévenin and Norton Equivalent Circuits

(a) Find the Thévenin and Norton equivalent circuits seen from outside of the box.



Solution:
To find the Thévenin and Norton equivalent circuits, we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.
For finding the open circuit voltage between the output ports, let us label the nodes as shown in the figure below.



First, let us begin by calculating the effective resistance between nodes A and GND. We have the $3\,\Omega$ resistor in parallel to the $4.5\,\Omega + 1.5\,\Omega$ resistance. This gives an equivalent resistance of

$$\frac{1}{\frac{1}{3\,\Omega} + \frac{1}{4.5\,\Omega+1.5\,\Omega}} = 2\,\Omega.$$

Then we see that we have a voltage divider from the positive terminal of the $10\,\text{V}$ supply. Voltage divider is made up of two resistances in series, where the resistances are $3\,\Omega$ and $2\,\Omega$. This gives the voltage at node A equal to

$$V_{\text{A}} = 10\,\text{V}\frac{2\,\Omega}{3\,\Omega + 2\,\Omega} = 4\,\text{V}$$

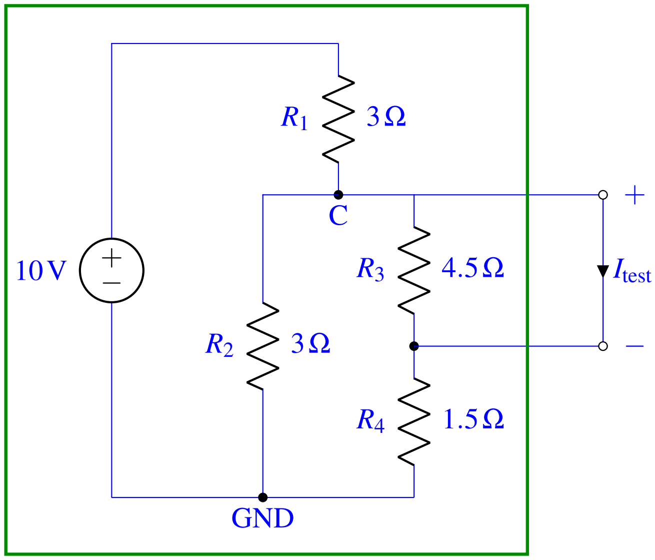
To find the voltage at node B, note that we have another voltage divider between nodes A and GND. Hence, we can find the voltage at node B as

$$V_{\text{A}}\frac{1.5\,\Omega}{4.5\,\Omega + 1.5\,\Omega} = 4\,\text{V} \cdot \frac{1}{4} = 1\,\text{V}$$

Hence the open circuit voltage seen between the output ports is equal to

$$\begin{aligned} V_{\text{open}} &= V_A - V_B \\ &= 4\,\text{V} - 1\,\text{V} \\ &= 3\,\text{V} \end{aligned}$$

Now we need to choose a V_{test} to find R_{Th}. For convenience, let us choose $V_{\text{test}} = V_{\text{open}}$. Then $I_{\text{test}} = I_{\text{sc}} = I_{\text{No}}$. This Now let us find the short circuit current flowing through the output ports. When doing this, we get the following circuit.



Note that when we short the output terminals, the voltages at the nodes change, this is why we changed the label of the node below the resistor R_1. Since there is a short circuit parallel to the resistor R_3, there will be no current flowing through it, hence we have

$$I_{\text{test}} = I_{R_4}$$

To find this current, let us find the equivalent resistance due to R_2 being connected parallel to R_4 when we short the output ports. We have $3\,\Omega$ parellel to $1.5\,\Omega$, which gives an equivalent resistance

$$\frac{1}{\frac{1}{3\,\Omega} + \frac{1}{1.5\,\Omega}} = 1\,\Omega$$

We again have a voltage divider between the positive side of the 10 V supply and the ground. Using this voltage divider, we calculate the voltage at node C as

$$V_C = 10\,\text{V}\frac{1\,\Omega}{3\,\Omega + 1\,\Omega} = 2.5\,\text{V}$$

Hence, we see that the voltage across the resistor R_4 is equal to 2.5 V. Using Ohm's law, we get

$$I_{R_4} = \frac{2.5\,\text{V}}{1.5\,\Omega} = \frac{5}{3}\,\text{A}$$

Since we have $I_{\text{test}} = I_{R_4}$, we have

$$I_{\text{test}} = I_{R_4}$$

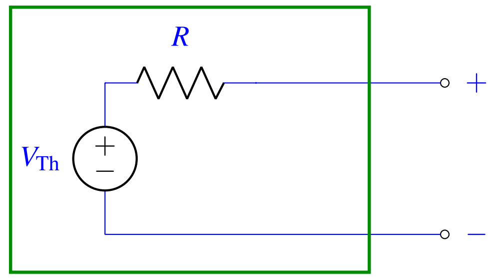
Summarizing the results, we have

$$V_{\text{test}} = 3\,\text{V}$$
$$I_{\text{test}} = \frac{5}{3}\,\text{A}$$

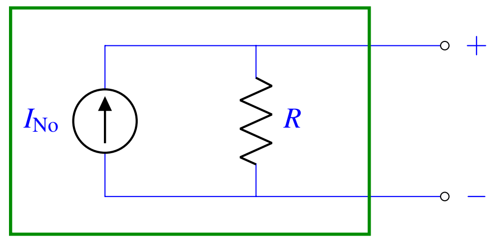
This gives

$$R_{\text{Th}} = \frac{V_{\text{test}}}{I_{\text{test}}} = \frac{9}{5}\,\Omega$$

Hence the Thévenin equivalent circuit is given by

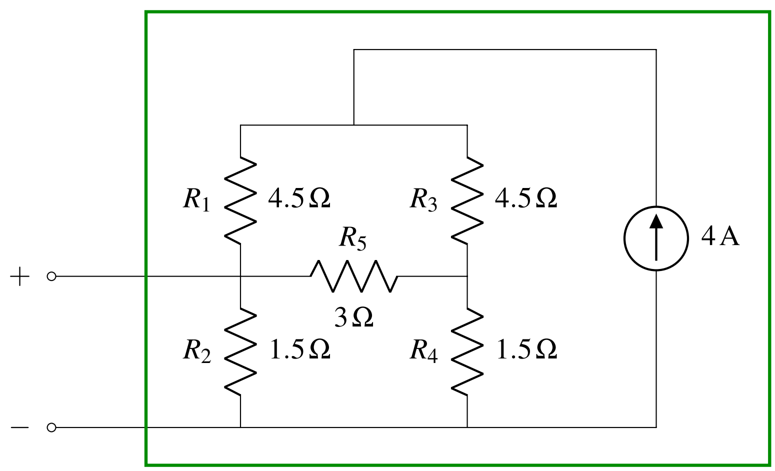


where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by



where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{test}}$.

(b) Find the Thévenin and Norton equivalent circuits seen from outside of the box.



Solution:

As with the previous part of this question, to find the Thévenin and Norton equivalent circuits we are going to find the open circuit voltage between the output ports and the current flowing through the output ports when the ports are shorted.

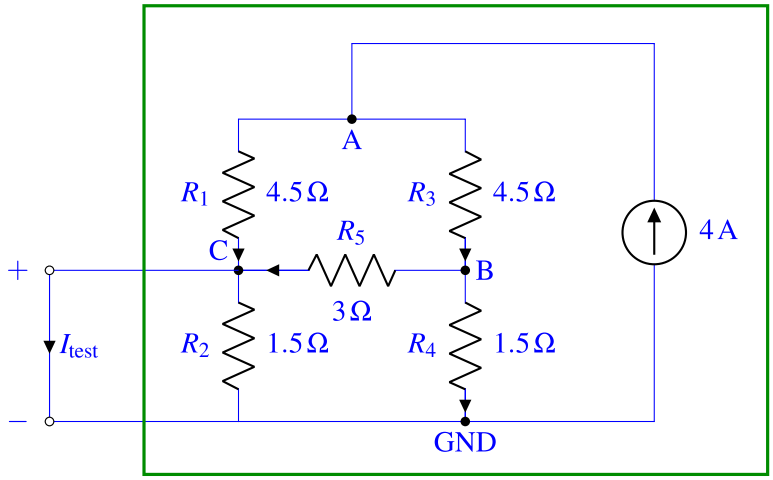
Let us first find the open circuit voltage between the output ports. We can use the symmetry in the circuit to see there will be no current flowing through R_5. Then, we have a current divider where each branch has the same resistance, hence the current 4 A divides equally between the left and right branches. Hence we have

$$I_{R_1} = I_{R_2} = I_{R_3} = I_{R_4} = 2\,\text{A}$$

This gives us the voltage across R_2, equivalently the open circuit voltage between the output terminals, equal to

$$V_{\text{open}} = 2\,\text{A} \times 1.5\,\Omega = 3\,\text{V}$$

Similar to part (a), we use the Thévenin voltage as V_{test} for convenience, since that sets I_{test} to the short circuit current I_{No}. Now let us find the short circuit current across the output terminals. Let us find this using nodal analysis on the resulting circuit when we short the output ports. To help do the analysis, let us label the nodes as shown in the figure below.



Now what are the unkown node voltages? We do not know the voltage at node A and B. On the other hand, because node C is connected by a short circuit to GND, we know its voltage is equal to the GND which we set as 0; hence voltage at node C is not an unkown. Next, because there is a short circuit across resistor R_2, there will be no current flowing through it.

Let us write KCL at the nodes

$$4\,\mathrm{A} = I_{R_1} + I_{R_3} \quad \text{(Node A)}$$
$$I_{R_3} = I_{R_4} + I_{R_5} \quad \text{(Node B)}$$
$$I_{\text{test}} = I_{R_1} + I_{R_5} \quad \text{(Node C)}$$

Now let us relate the currents I_{R_1}, I_{R_2}, I_{R_3}, I_{R_4} and I_{R_5} to node voltages using Ohm's law. We have

$$I_{R_1} = \frac{V_A - V_C}{R_1} = \frac{V_A}{4.5\,\Omega}$$

since $V_C = 0\,\mathrm{V}$ because it is connected to the ground by short circuit. Furthermore, we have

$$I_{R_2} = 0\,\mathrm{A},$$
$$I_{R_3} = \frac{V_A - V_B}{R_3} = \frac{V_A - V_B}{4.5\,\Omega}$$
$$I_{R_4} = \frac{V_B}{R_4} = \frac{V_B}{1.5\,\Omega},$$
$$I_{R_5} = \frac{V_B - V_C}{R_5} = \frac{V_B}{3\,\Omega}$$

Plugging these into the first two KCL equations, we get

$$4\,\mathrm{A} = \frac{V_A}{4.5\,\Omega} + \frac{V_A - V_B}{4.5\,\Omega}$$
$$\frac{V_A - V_B}{4.5\,\Omega} = \frac{V_B}{1.5\,\Omega} + \frac{V_B}{3\,\Omega}$$

These equations are solved by

$$V_A = 9.9\,\text{V},$$
$$V_B = 1.8\,\text{V}$$

Using the KCL at node C, we get

$$\begin{aligned} I_{\text{test}} &= I_{R_1} + I_{R_5} \\ &= \frac{V_A}{4.5\,\Omega} + \frac{V_B}{3\,\Omega} \\ &= \frac{9.9}{4.5\,\Omega} + \frac{1.8}{3\,\Omega} \\ &= 2.8\,\text{A} \end{aligned}$$

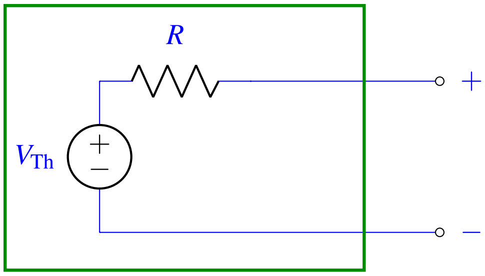
Summarizing the results, we have

$$V_{\text{open}} = 3\,\text{V}$$
$$I_{\text{test}} = 2.8\,\text{A}$$

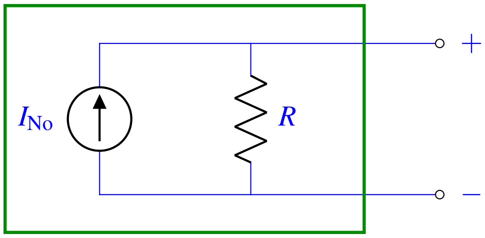
This gives

$$\begin{aligned} R_{\text{Th}} &= \frac{V_{\text{test}}}{I_{\text{test}}} \\ &= \frac{15}{14}\,\Omega \end{aligned}$$

Hence the Thévenin equivalent circuit is given by



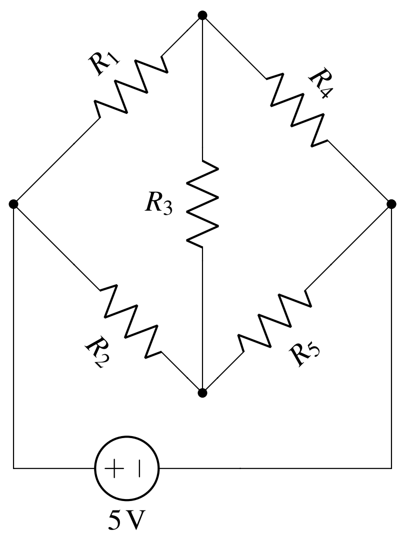
where $R = R_{\text{Th}}$ and $V_{\text{Th}} = V_{\text{open}}$, and the Norton equivalent circuit is given by



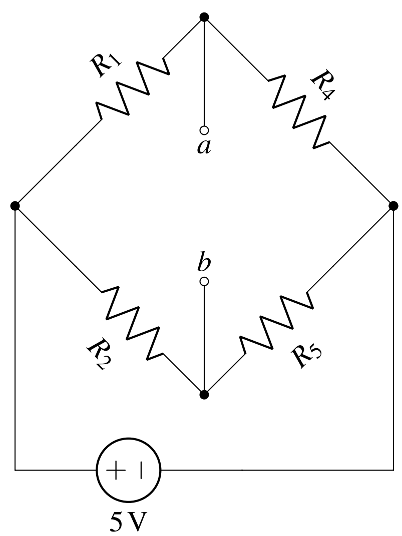
where $R = R_{\text{Th}}$ and $I_{\text{No}} = I_{\text{test}}$.

2. Wheatstone Bridge

Thévenin equivalence is a powerful technique we can use to solve the Wheatstone bridge circuit shown below. This circuit is used in many sensor application where a sensing element is the "bridge" resistor, R_3. It is often useful to find the current through the bridge resistor or the voltage across the bridge resistor. Intuitively, knowing I_{R_3} or V_{R_3} allows us to solve the rest of the circuit. In this problem, we want to find the current I_{R_3} flowing through the bridge resistor R_3.



(a) First, let's remove the bridge resistor R_3. Calculate the Thévenin equivalent voltage V_{th} between the two terminals a and b, for the circuit shown below, where the bridge resistor has been removed.



Solution:

Notice in the above circuit that there are two voltage dividers, so we can calculate v_a and v_b quickly.

$$v_a = \frac{R_4}{R_1 + R_4} \cdot 5\,\text{V}$$
$$v_b = \frac{R_5}{R_2 + R_5} \cdot 5\,\text{V}$$

Thus, the Thévenin voltage is simply the difference between the two voltages: $V_{th} = v_a - v_b = \left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V}$.

(b) Is the Thévenin voltage V_{th} you found in part (a) equal to the actual voltage V_{R_3} across the bridge resistor? Why or why not?

Solution:

No, the Thévenin voltage we found in part (a) is the open-circuit voltage. If we add R_3 back into the original circuit, R_3 would load the other resistors (or, equivalently, the Thévenin resistance), so the Thévenin voltage is not equal to the actual voltage across the bridge resistor.

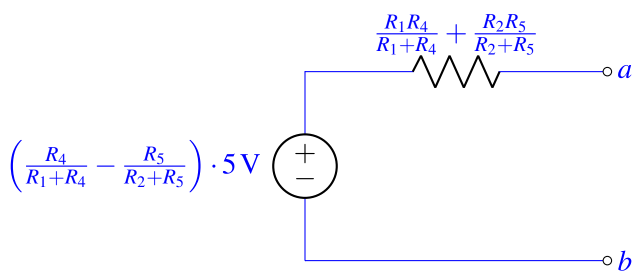
(c) Find the Thévenin resistance R_{th} between the two terminals a and b for the above circuit. Draw the Thévenin equivalent between the terminals a and b for the circuit above.

Solution:

We find the Thévenin resistance by replacing the voltage source with a short and calculating the resistance between the two terminals a and b.

$$\begin{aligned} R_{th} &= (R_1 \parallel R_4) + (R_2 \parallel R_5), \text{ where } \parallel \text{ denotes the parallel operator.} \\ &= \frac{R_1 R_4}{R_1 + R_4} + \frac{R_2 R_5}{R_2 + R_5} \end{aligned}$$

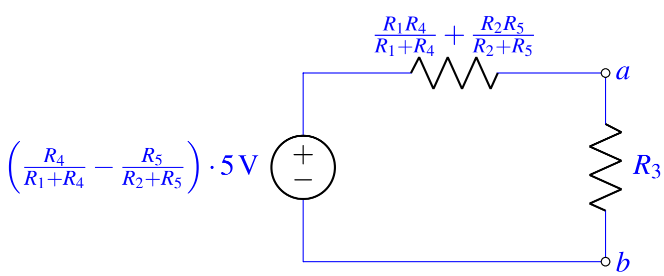
Using $V_{th} = \left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V}$ from part (a), we can construct the Thévenin equivalent circuit.



(d) With this equivalent circuit, calculate the current I_{R_3} through the bridge resistor and the voltage V_{R_3} across the bridge resistor.

Solution:

Using the equivalent circuit, we now add R_3 back in.

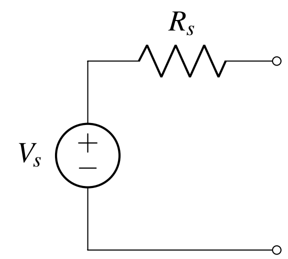


$$I_{R_3} = \frac{\left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V}}{R_3 + \frac{R_1R_4}{R_1+R_4} + \frac{R_2R_5}{R_2+R_5}}$$

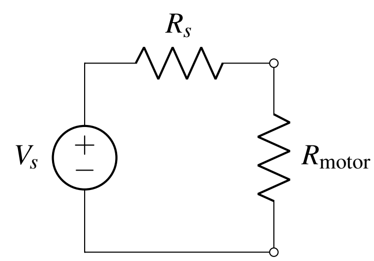
$$V_{R_3} = I_{R_3} R_3 = \frac{\left(\frac{R_4}{R_1+R_4} - \frac{R_5}{R_2+R_5}\right) \cdot 5\,\text{V} \cdot R_3}{R_3 + \frac{R_1R_4}{R_1+R_4} + \frac{R_2R_5}{R_2+R_5}}$$

3. Maximum Horsepower

You are an engineer working on an electric car. Your job is to design a motor to be used the car. Specifically you are designing the resistance of this motor. The battery used by this car has some series resistance, and it is modeled by the circuit shown below.



You attach your motor to the battery as shown below.



(a) Calculate the power P_s delivered by the voltage source in terms of V_s, R_s, and R_{motor}.

Solution:

$$I_s = \frac{V_s}{R_s + R_{\text{motor}}}$$

$$P_s = -I_s V_s = -\frac{V_s^2}{R_s + R_{\text{motor}}}$$

Using passive sign convention, we obtain the power dissipated by the source. This is a negative quantity, implying that the source is delivering power. The power delivered by the source is then $\frac{V_s^2}{R_s + R_{\text{motor}}}$.

(b) Now calculate the power P_{motor} dissipated by the load resistor in terms of V_s, R_s, and R_{motor}.

Solution:

$$V_{\text{motor}} = \frac{R_{\text{motor}}}{R_s + R_{\text{motor}}} \cdot V_s$$

$$P_{\text{motor}} = \frac{V_{\text{motor}}^2}{R_{\text{motor}}} = \frac{R_{\text{motor}}}{(R_s + R_{\text{motor}})^2} \cdot V_s^2$$

(c) Suppose we wanted to maximize the power dissipated across the load. Find the optimal value for R_{motor} in terms of R_s.

Hint: Use calculus.

Solution:

$$P_{\text{motor}} = \frac{R_{\text{motor}}}{(R_s + R_{\text{motor}})^2} \cdot V_s^2$$

$$\frac{dP_{\text{motor}}}{dR_{\text{motor}}} = \frac{(R_s + R_{\text{motor}})^2 - 2R_{\text{motor}}(R_s + R_{\text{motor}})}{(R_s + R_{\text{motor}})^4} \cdot V_s^2 = 0$$

$$R_s^2 - R_{\text{motor}}^2 = 0$$

$$R_{\text{motor}}^2 = R_s^2$$

$$R_{\text{motor}} = R_s$$

(d) Now you've swiched teams to designing the battery. Your job is now to pick the optimal R_s for maximizing the power delivered to the motor. What value of R_s should you pick?

Hint: Don't use calculus.

Solution:

To maximize the power delivered to the motor, we need to maximize the voltage across the motor. This is done when there is no R_s. Thus, the optimal value of R_s is $0\,\Omega$.

4. Digital to Analog Converter (DAC)

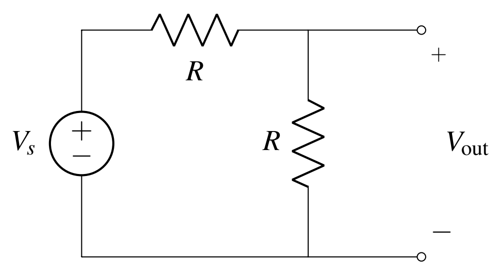
For some outputs, such as audio applications, we need to produce an analog output, or a continuous voltage from 0 to V_s. These analog voltages must be produced from digital voltages, that is sources, that can only be V_s or 0. A circuit that does this is known as a Digital to Analog Converter. It takes a binary representation of a number and turns it into an analog volage.

The output of a DAC can be represented with the equation shown below:

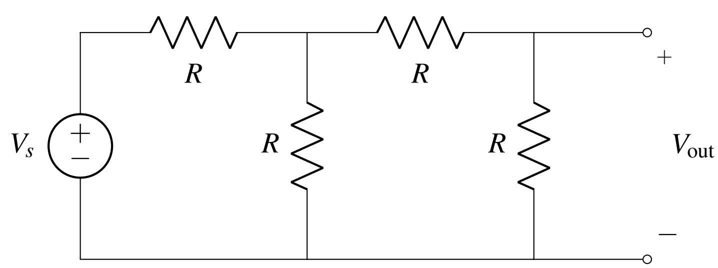
$$V_{\text{out}} = V_s \sum_{n=0}^{N} \frac{1}{2^n} \cdot b_n$$

where each binary digit b_n is multiplied by $\frac{1}{2^n}$.

(a) We know how to take an input voltage and divide it by 2:



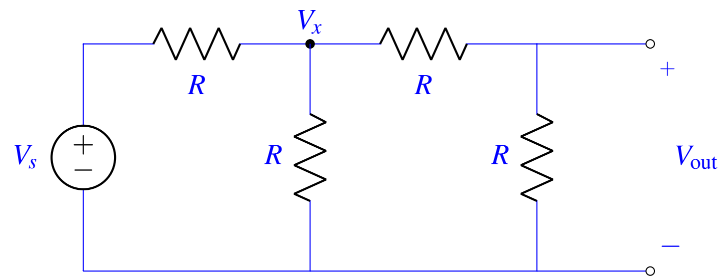
To divide by larger powers of two, we might hope to just "cascade" the above voltage divider. For example, consider:



Calculate V_{out} in the above circut. Is $V_{\text{out}} = \frac{1}{4}V_s$?

Solution:

We first find the potential V_x.

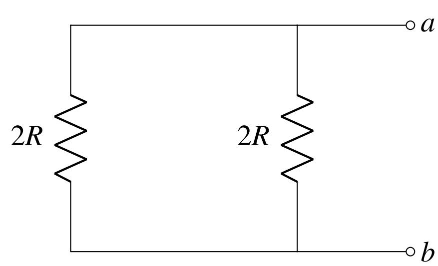


$$V_x = \frac{R \parallel 2R}{R + R \parallel 2R} V_s = \frac{\frac{2}{3}R}{R + \frac{2}{3}R} V_s = \frac{2}{5} V_s$$

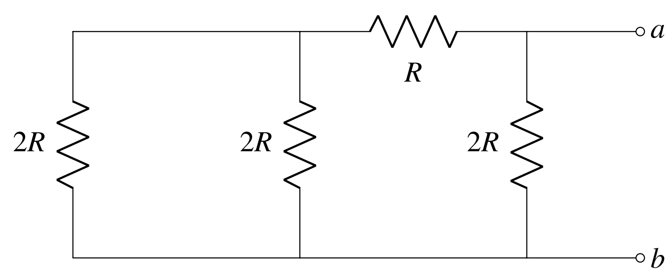
$$V_{\text{out}} = \frac{R}{R+R} V_x = \frac{1}{2} \cdot \frac{2}{5} V_s = \frac{1}{5} V_s \neq \frac{1}{4} V_s$$

(b) The R-$2R$ ladder, shown below, has a very nice property. For each of the circuits shown below, find the equivalent resistance looking in from points a and b. Do you see a pattern?

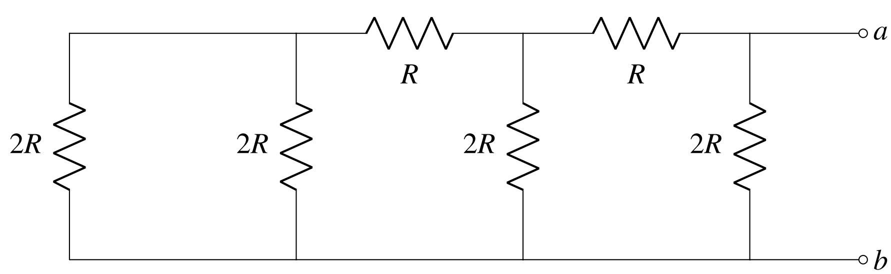
i.



ii.



iii.

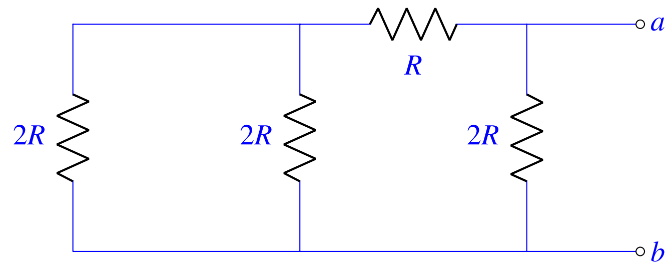


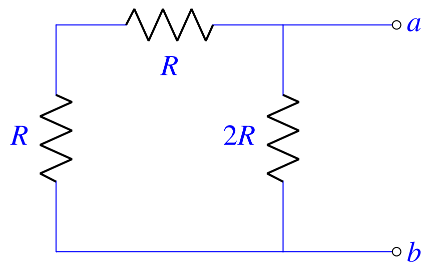
Solution:

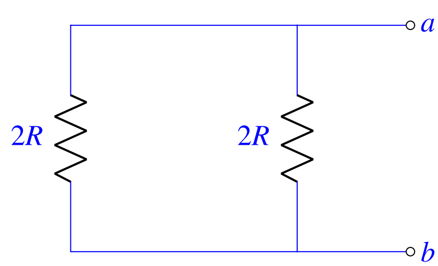
i.

$$R_{eq} = 2R \parallel 2R = R$$

ii. We find the equivalent resistance for the resistors from left to right.

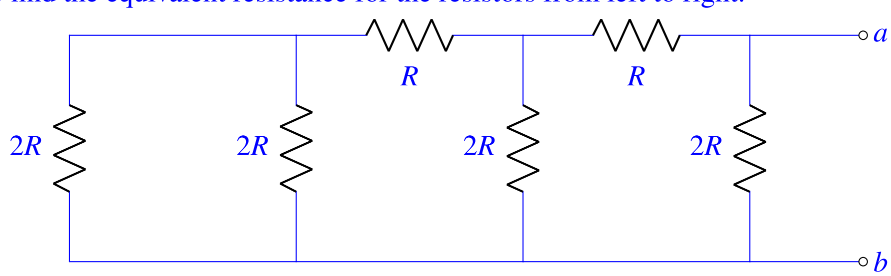


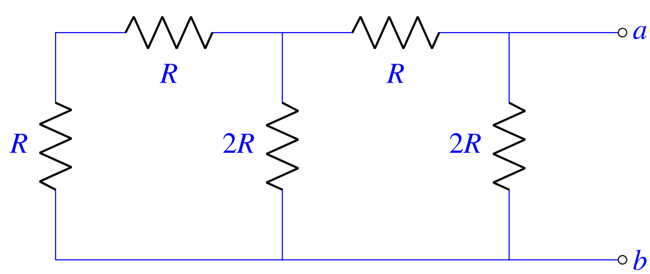


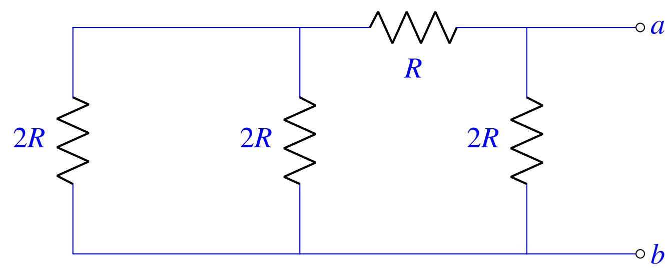


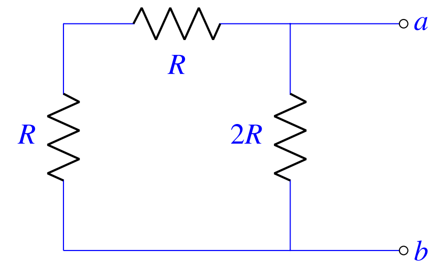
$$R_{eq} = 2R \parallel 2R = R$$

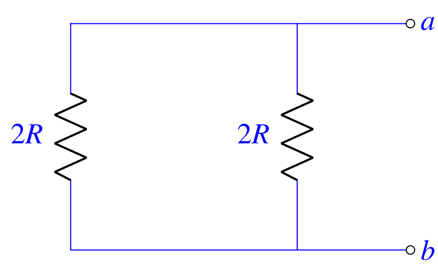
iii. Again, we find the equivalent resistance for the resistors from left to right.







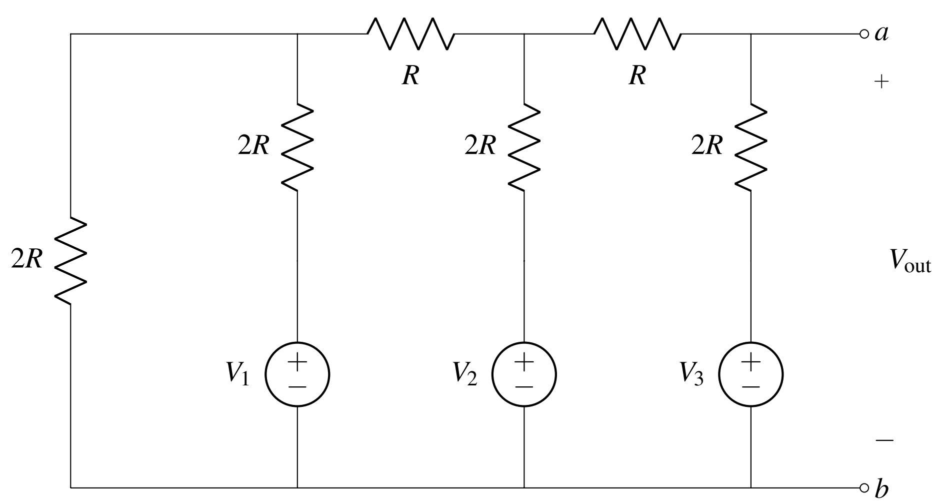




$$R_{eq} = 2R \parallel 2R = R$$

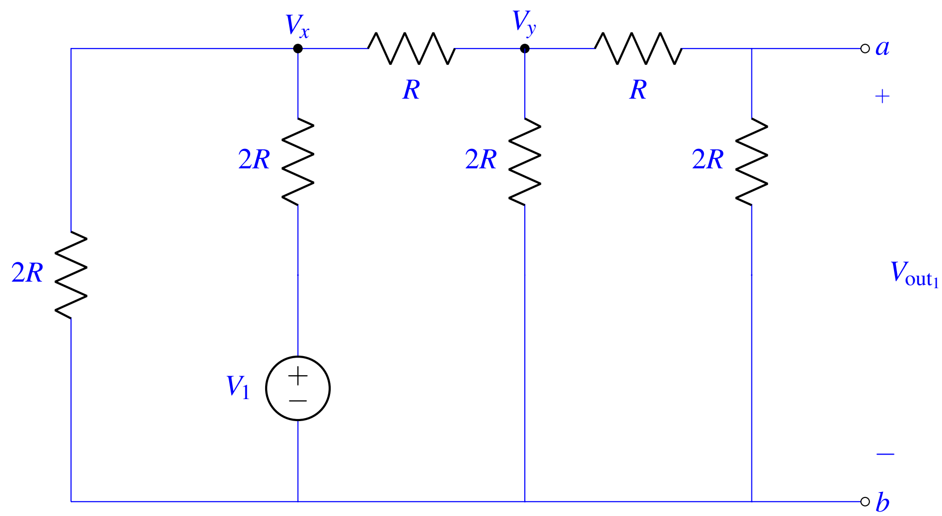
The equivalent resistance is always $R_{eq} = R$.

(c) The following circuit is an R-$2R$ DAC. To understand its functionality, use superposition to find V_{out} in terms of each V_k in the circuit.

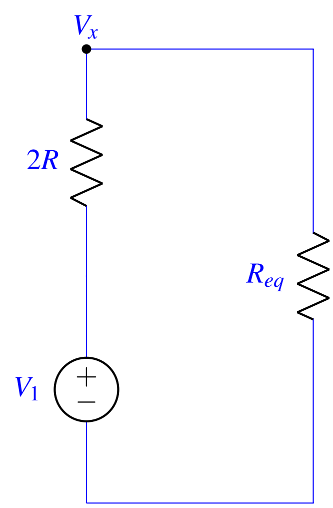


Solution:

V_1:



We first find the potential V_x. To do this, we can simplify the circuit.



$$R_{eq} = 2R \parallel (R + (2R \parallel (R + 2R))) = \frac{22}{21}R$$

We can then find V_x using the voltage divider formula.

$$V_x = \frac{\frac{22}{21}R}{2R + \frac{22}{21}R} V_1 = \frac{11}{32} V_1$$

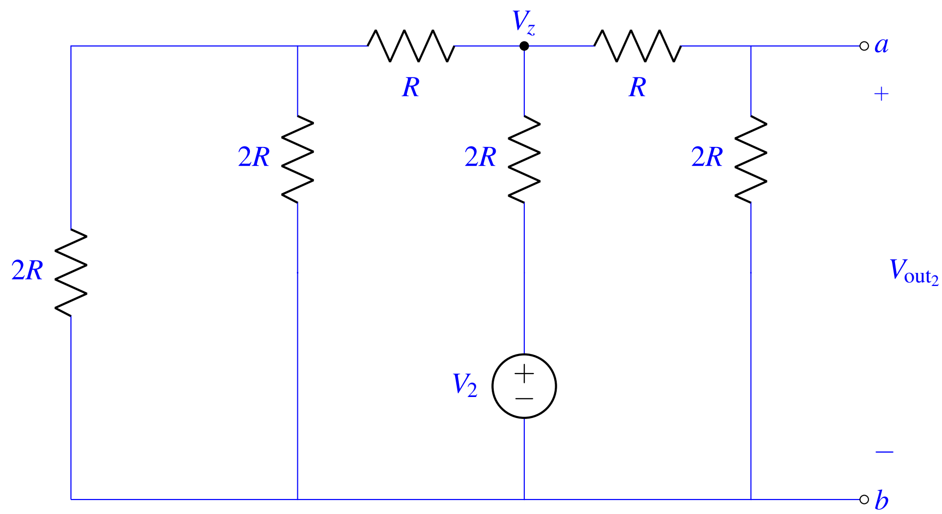
Similarly, we use the voltage divider formula to find V_y in terms of V_x.

$$V_y = \frac{2R \parallel (R + 2R)}{R + 2R \parallel (R + 2R)} V_x = \frac{\frac{6}{5}R}{R + \frac{6}{5}R} V_x = \frac{6}{11} \cdot \frac{11}{32} V_1 = \frac{3}{16} V_1$$

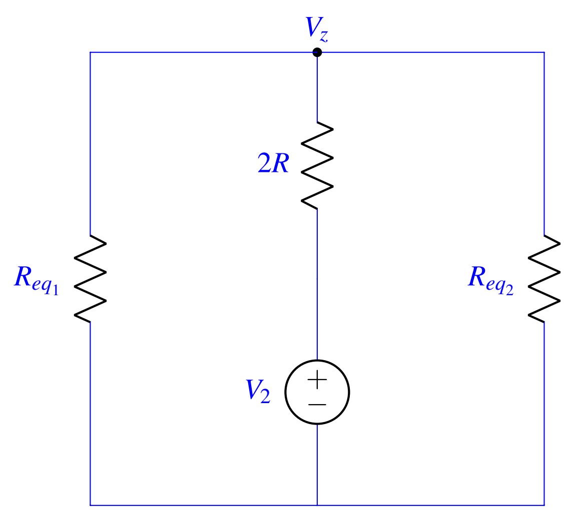
Applying the voltage divider formula again gives us V_{out_1}.

$$V_{\text{out}_1} = \frac{2R}{R + 2R} V_y = \frac{2}{3} \cdot \frac{3}{16} V_1 = \frac{1}{8} V_1$$

V_2:



We first find the potential V_z. To do this, we can simplify the circuit.



$$R_{eq_1} = R + (2R \parallel 2R) = R + R = 2R$$

$$R_{eq_2} = R + 2R = 3R$$

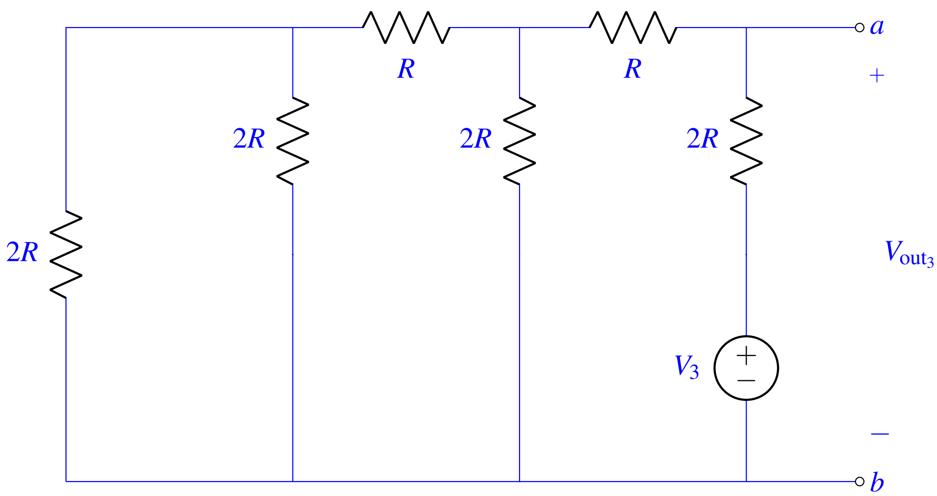
We can then find V_z using the voltage divider formula.

$$V_z = \frac{2R \parallel 3R}{2R + (2R \parallel 3R)} V_2 = \frac{\frac{6}{5}R}{2R + \frac{6}{5}R} V_2 = \frac{3}{8} V_2$$

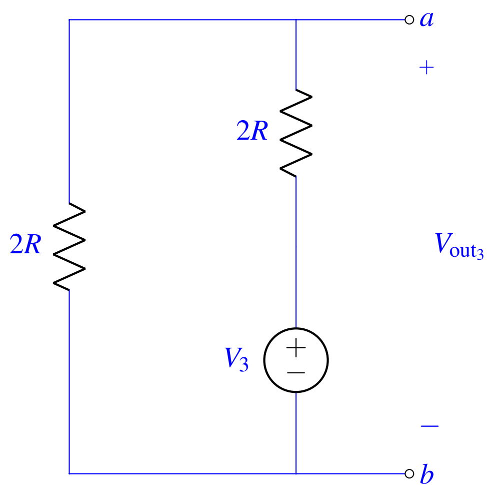
Applying the voltage divider formula again gives us V_{out_2}.

$$V_{\text{out}_2} = \frac{2R}{R + 2R} V_z = \frac{2}{3} \cdot \frac{3}{8} V_2 = \frac{1}{4} V_2$$

V_3:



We can simplify this circuit.



$$V_{\text{out}_3} = \frac{2R}{2R+2R} V_3 = \frac{1}{2} V_3$$

$$V_{\text{out}} = V_{\text{out}_1} + V_{\text{out}_2} + V_{\text{out}_3} = \frac{1}{8} V_1 + \frac{1}{4} V_2 + \frac{1}{2} V_3$$

(d) We've now designed a 3-bit R-$2R$ DAC. What is the output voltage V_{out} if $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$?

Solution:

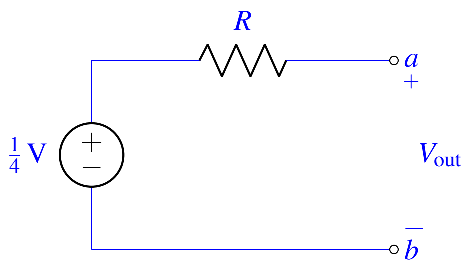
$$V_{\text{out}} = \frac{1}{8} \cdot 0\,\text{V} + \frac{1}{4} \cdot 1\,\text{V} + \frac{1}{2} \cdot 0\,\text{V} = \frac{1}{4}\,\text{V}$$

(e) Draw the Thévenin equivalent of the above circuit, looking in from the terminals a and b with $V_2 = 1\,\text{V}$ and $V_1 = V_3 = 0\,\text{V}$.

Solution:

$$V_{th} = \frac{1}{4}\,\text{V}$$

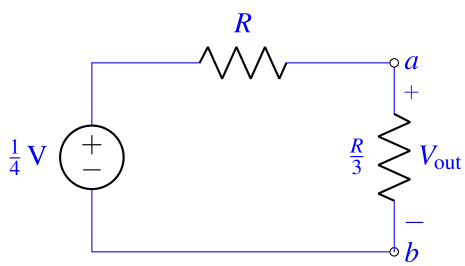
$$R_{th} = R$$



(f) Suppose that we now attach a speaker to the DAC with a resistance of $\frac{R}{3}$. Why is the voltage across the speaker lower than what we computed in part (d)? What is the actual output voltage?

Solution:

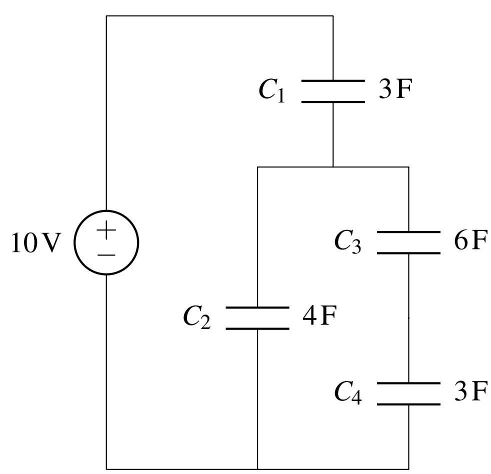
Attaching the speaker will load the DAC. The speaker draws some current from the DAC, so the voltage across the output will be lower than expected. We can use the Thévenin equivalent circuit to calculate the actual output voltage.



$$V_{\text{out}} = \frac{\frac{R}{3}}{R + \frac{R}{3}} \cdot \frac{1}{4}\,\text{V} = \frac{1}{4} \cdot \frac{1}{4}\,\text{V} = \frac{1}{16}\,\text{V}$$

5. Mechanical Circuits with Capacitors and Resistors

Find the voltages across and currents flowing through all of the capacitors at steady state.



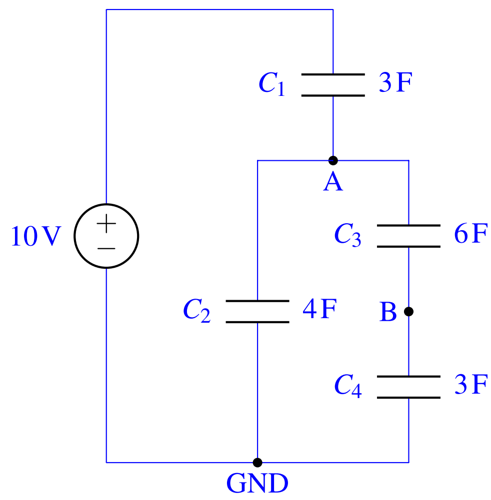
Solution:

For a capacitor C_k, let us denote the voltage across it by v_{C_k}, the current flowing through it by i_{C_k}, and its charge by Q_{C_k}. In steady state (that is, after the current has running for a very long time), direct current

(DC) capacitors act as open circuits. Hence, we see that there is no current flowing through the capacitors, that is,

$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0\,\text{A}.$$

For finding the voltages across the capacitors, let us label nodes on the circuit as shown in the following figure.



We are going to use the following four properties to find the voltages across the capacitors:

(a) Charge is always conserved.

(b) The charge Q stored in a capacitor is given by the equation $Q = CV$.

(c) The charges across series capacitors are equal to each other.

(d) The voltage across parallel capacitors is equal.

As an example use of property (c), we have the charge on the capacitor C_3 equal to the charge on the capacitor C_4.

Let us start by writing the equation for conservation of charge at node A:

$$Q_{C_1} = Q_{C_2} + Q_{C_3}$$

By property (b), that is, $Q = CV$, we can equivalently write this eqaution for charge conservation in terms of node voltages as

$$(10\,\text{V} - v_A)3\,\text{F} = v_A 4\,\text{F} + (v_A - v_B)6\,\text{F},$$

which, after simplifiying the equation, gives

$$30\,\text{V} = 13v_A - 6v_B. \tag{1}$$

Let us then write the charge conservation equation at node B; we have

$$Q_{C_3} = Q_{C_4}.$$

As before, we can write this charge conservation equation in terms of the node voltages as

$$(v_A - v_B)6\,\text{F} = v_B 3\,\text{F},$$

which, after simplification, gives

$$2v_A = 3v_B. \tag{2}$$

Equations 1 and 2 give us two linearly independent equations in two unknowns. Solving the system, we get

$$v_A = \frac{10}{3}\,\mathrm{V},$$
$$v_B = \frac{20}{9}\,\mathrm{V}.$$

Using the node voltages, we can calculate the voltages across the capacitors as

$$v_{C_1} = 10\,\mathrm{V} - v_A = \frac{20}{3}\,\mathrm{V},$$
$$v_{C_2} = v_A = \frac{10}{3}\,\mathrm{V},$$
$$v_{C_3} = v_A - v_B = \frac{10}{9}\,\mathrm{V},$$
$$v_{C_4} = v_B = \frac{20}{9}\,\mathrm{V}.$$

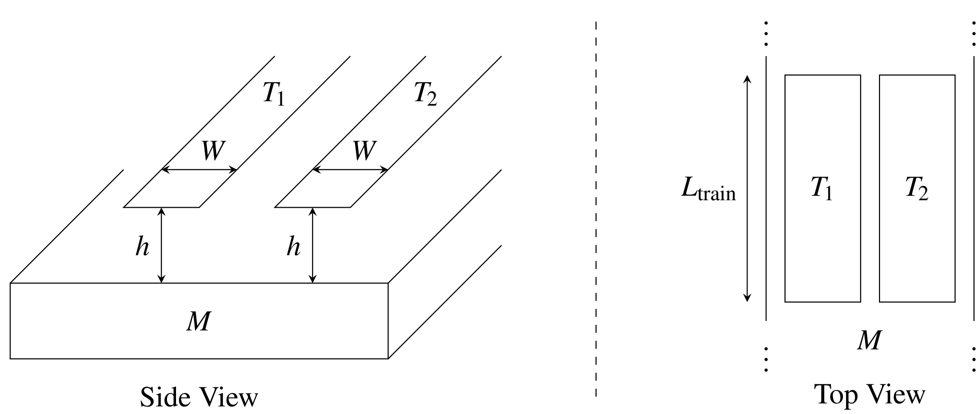
We write the currents across the capacitors again here for reader's convenience:

$$i_{C_1} = i_{C_2} = i_{C_3} = i_{C_4} = 0\,\mathrm{A}$$

6. Maglev Train Height Control System

One of the fastest forms of land transportation are trains that actually travel slightly elevated from ground using magnetic levitation (or "maglev" for short). Ensuring that the train stays at a relatively constant height above its "tracks" (the tracks in this case are what provide the force to levitate the train and propel it forward) is critical to both the safety and fuel efficiency of the train. In this problem, we'll explore how the maglev trains use capacitors to keep them elevated. (Note that real maglev trains may use completely different and much more sophisticated techniques to perform this function, so if you e.g. get a contract to build such a train, you'll probably want to do more research on the subject.)

(a) As shown below, let's imagine that all along the bottom of the train, we put two parallel strips of metal (T_1, T_2), and that on the ground below the train (perhaps as part of the track), we have one solid piece of metal (M).



Assuming that the entire train is at a uniform height above the track and ignoring any fringing fields (i.e., we can use the simple equations developed in lecture to model the capacitance), as a function of L_{train} (the length of the train), W (the width of T_1/T_2), and h (the height of the train off of the track), what is the capacitance between T_1 and M? How about the capacitance between T_2 and M?

Solution:

The distance between the plates (T_1 & M or T_2 & M) is h. The area of plate for the parallel plate capacitor is $A = WL_{\text{train}}$. Using the formula for capacitance of a parallel plate capacitor, we get:

$$C = \frac{\varepsilon A}{d}$$

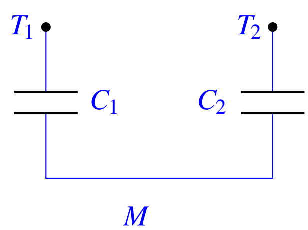
$$C_1 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } T_1 \text{ and M)}$$

$$C_2 = \frac{\varepsilon W L_{\text{train}}}{h} \text{ (Capacitance between } T_2 \text{ and M)}$$

(b) Any circuit on the train can only make direct contact at T_1 and T_2. Thus, to detect the height of the train, it would only be able to measure the equivalent capacitance between T_1 and T_2. Draw a circuit model showing how the capacitors between T_1 and M and between T_2 and M are connected to each other.

Solution:

The capacitors C_1 and C_2 are in series. To realize this, let's consider the train circuit that is in contact with T_1 and T_2. If there is current entering plate T_1, the same current has to exit plate T_2. Thus, the circuit can be modeled as follows:



(c) Using the same parameters as in part (a), provide an expression for the equivalent capacitance between T_1 and T_2.

Solution:

Since the two capacitors are in series, the equivalent capacitance between T_1 and T_2 is given by:

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2}$$

Thus, we get

$$\frac{1}{C_{\text{eq}}} = \frac{h}{\varepsilon W L_{\text{train}}} + \frac{h}{\varepsilon W L_{\text{train}}}$$

$$C_{\text{eq}} = \frac{\varepsilon W L_{\text{train}}}{2h}$$

(d) Let's assume that instead of just detecting the height (by measuring the equivalent capacitance between T_1 and T_2), we also want to control it. Let's assume that the device we use to control the height takes in only one of only two commands: increase the height, or decrease the height. In particular, this device is controlled by an input voltage. If that voltage is greater than 2.5 V, it will push the train higher above the track, and if it is less than 2.5 V, it will let the train move down closer to the track.

Assuming that the train is 100 m long ($L_{train} = 100\,\text{m}$) and that the T_1/T_2 metals are each 1 cm wide ($W = 1\,\text{cm}$), design a circuit that will feed a voltage into the control device to make the train levitate 1 cm above the track. Be sure to show how your circuit is connected to T_1 and T_2, and be as specific as possible in terms of the component values you would use. You can use any combination of switches, voltage sources, current sources, resistors, and capacitors that you would like to implement this circuit.

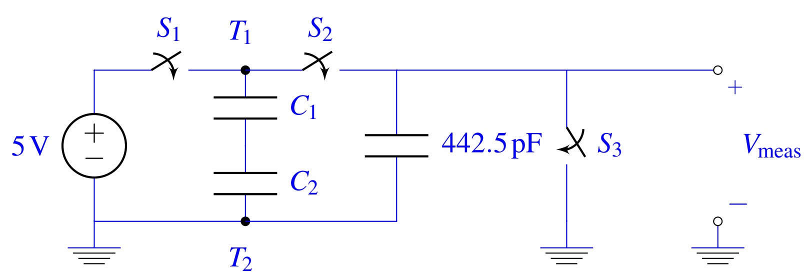
Solution:

The equivalent capacitance between T_1 and T_2 is inversely proportional to the height of the train. We are only allowed to measure the equivalent capacitance between T_1 and T_2, we can measure what the capacitance is to figure out what h is. Based on whether h is lower or higher, we can construct a circuit using switches and capacitors (to figure out whether capacitance and thus height is above or below the desired height) whose output can then drive the height controller mechanism.

First, let's figure out what C_{eq} should be at the desired height:

$$C_{eq} = 8.85\frac{\text{pF}}{\text{m}} \cdot \frac{100\,\text{m} \cdot 10^{-2}\text{m}}{2 \cdot 10^{-2}\text{m}} = 442.5\,\text{pF}$$

Let's use a circuit similar to the one in the touchscreen lab to measure C_{eq}. (Note that there are many other circuits that can measure the capacitance as well. Any consistent and functioning solution will receive full credit.)

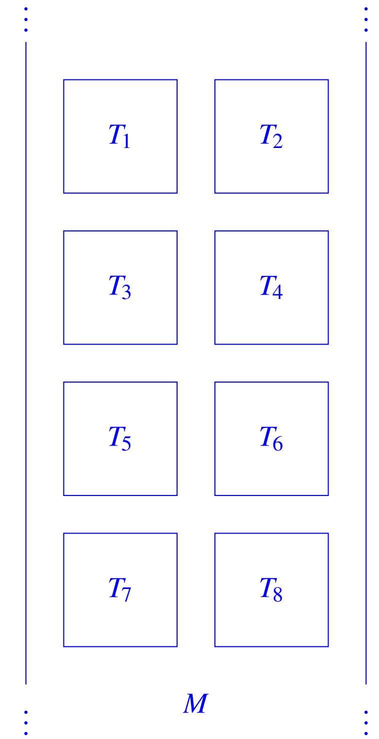


442.5 pF was chosen for the fixed capacitor, so that V_{meas} will be around 2.5 V.

If the train is higher than it is supposed to be (which is 1 cm), then $C_{eq} < 442.5\,\text{pF}$. After charge sharing with the fixed 442.5 pF capacitor, V_{meas} will therefore be $< 2.5\,\text{V}$, so the control device moves the train down. Similarly, if the train is lower than it is supposed to be then $V_{meas} > 2.5\,\text{V}$, so the control device moves the train up.

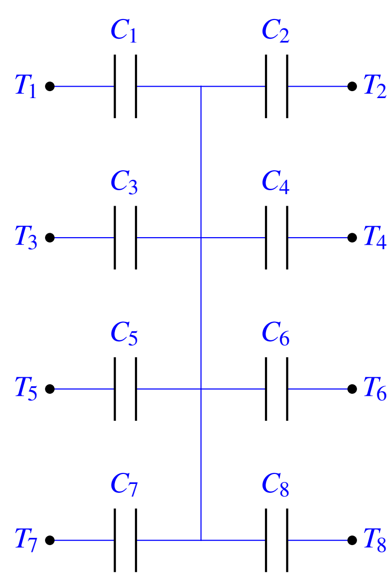
(e) So far we've assumed that the height of the train off of the track is uniform along its entire length, but in practice, this may not be the case. Suggest and sketch a modification to the basic sensor design (i.e., the two strips of metal T_1/T_2 along the entire bottom of the train) that would allow you to measure the height at the train at 4 different locations.

Solution:



Top View

One important thing to note about this circuit is that it works only if extra care is taken during the capacitance measurement circuit. The equivalent model for this is:



Therefore, the circuit needs separate switches on each T, so that you can measure the capacitance between only two terminals (like T_1 & T_2) and so that the effect of other capacitors is nullified.

7. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.