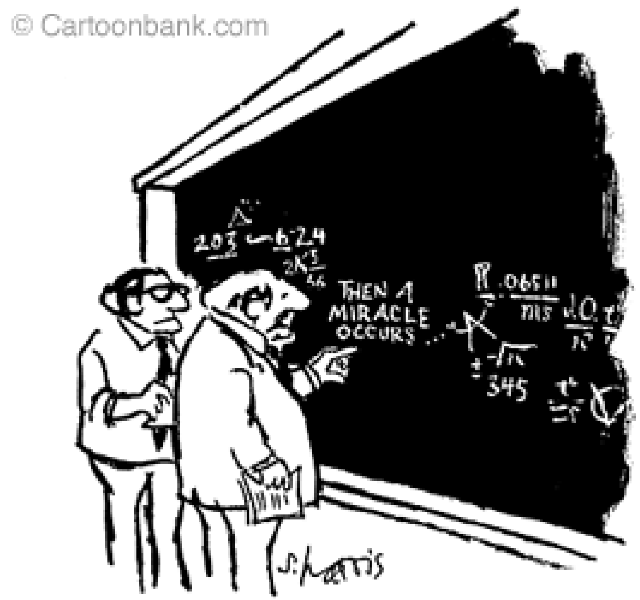


"I think you should be more explicit here in step two."

EE16A

Designing Information Devices and Systems I

Last time:

$$\vec{\mathsf{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\mathsf{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \boxed{A_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

Element 2n of the matrix

diagonal

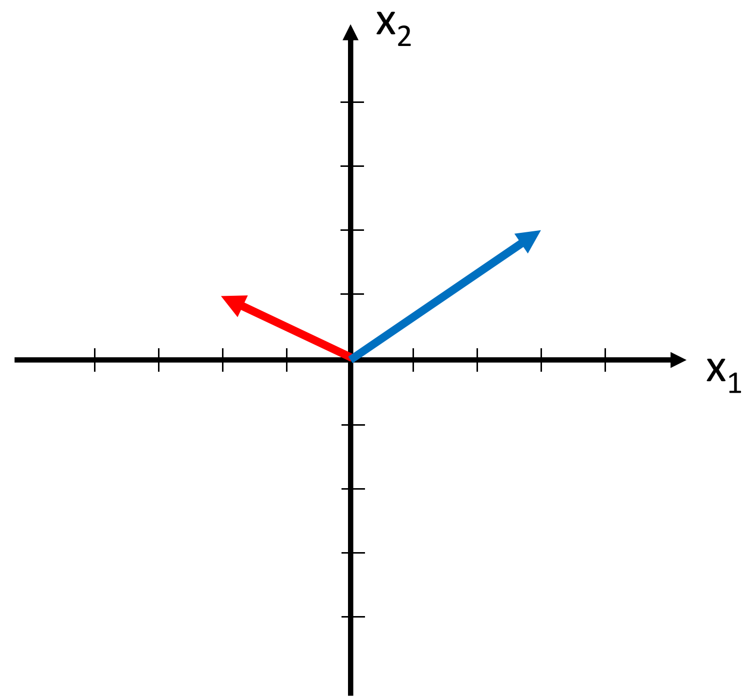
A **vector** is an array of numbers

A **matrix** is a rectangular arrray of numbers

Drawing these vectors graphically

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

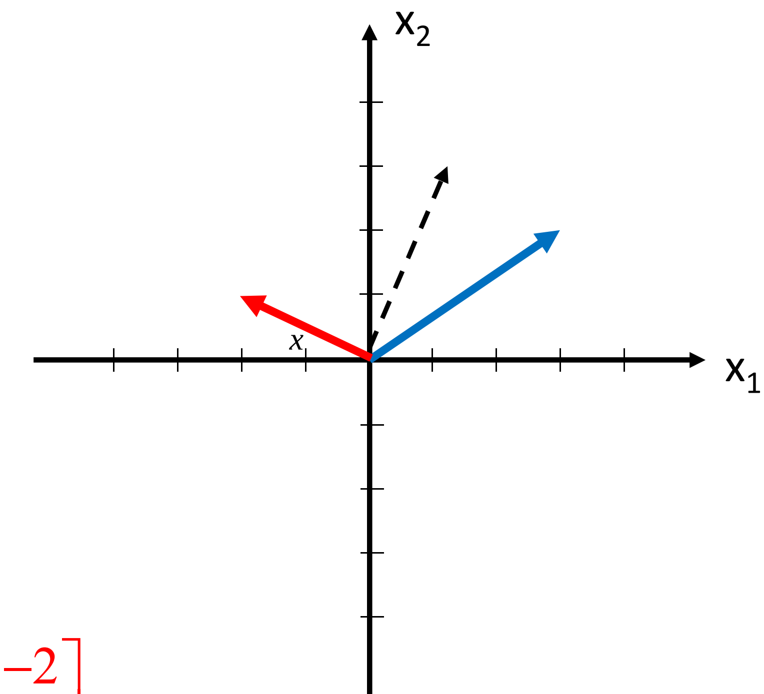
$$\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



What is the sum of the two vectors?

$$\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$

$$\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$



$$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2 \\ 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

To add vectors, add each corresponding element!

Draw it graphically

Does adding vectors $\vec{x}_1 + \vec{x}_2 = \vec{x}_2 + \vec{x}_1$?

yes.

Which of these apply?

- ✓ Commutativity: $\vec{x}+\vec{y}=\vec{y}+\vec{x}$
- ✓ Associativity: $\left(\vec{x}+\vec{y}\right)+\vec{z}=\vec{x}+\left(\vec{y}+\vec{z}\right)$
- ✓ Additive identity: $\vec{x}+\vec{0}=\vec{x}$
- ✓ Additive inverse: $\vec{x}+(-\vec{x})=\vec{0}$

Adding matrices

$$\vec{x}_1 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$\vec{x}_1 + \vec{x}_2 = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 1+0 \\ 3+3 & 4+2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}$$

To add matrices, add each corresponding element!

What if they are not same dimensions?

Then you cannot add them.

Vector transpose

$$\vec{\mathrm{x}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \Rightarrow \vec{\mathrm{x}}^T = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_N \end{bmatrix}$$

T for Transpose!

Matrix transpose

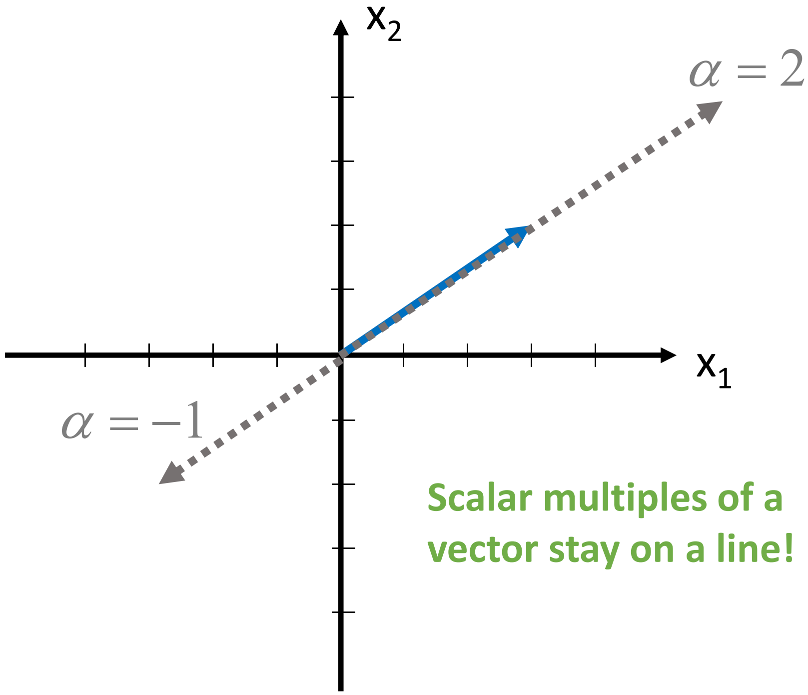
Swaps the rows with the columns

$$\vec{x} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \longrightarrow \vec{x}^T = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \longrightarrow \vec{x}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Scaling vectors

$$\vec{\mathbf{x}}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in \mathbb{R}^2$$



What is $\alpha\vec{\mathbf{x}}_1$?

$$\alpha\vec{\mathbf{x}}_1 = \alpha\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha 3 \\ \alpha 2 \end{bmatrix}$$

A vector multiplied by a scalar multiplies all elements of the vector by the scalar.

Multiplying matrices/vectors

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{n \times p} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}}_{m \times p}$$

Must be same!

Multiplying a vector by a vector

$\underset{m\times n}{A}\ \underset{n\times p}{B} = \underset{m\times p}{C}$ $\Rightarrow$ $\therefore$ a row vector ~~can only be~~ multiplied by a column vector

these need to be same!

$\underset{1\times n}{[\ \ ]}\ \underset{n\times 1}{[\ ]} = \underset{1\times 1}{\square}$

$$\vec{y}^T\vec{x} = \underset{1\times n}{\begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_n \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots y_nx_n}_{\substack{\text{scalar} \\ 1\times 1}}$$

Also known as "inner product" or "dot product"

$\boxed{\text{Is that same as } \vec{x}\vec{y}^T\text{?}}$ No. NOT commutative

Let's try it!

$$\vec{x}\vec{y}^T = \underset{n\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \underset{1\times m}{\begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}} = \underset{n\times m}{\begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_m \\ x_2y_1 & x_2y_2 & \cdots & x_2y_m \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_m \end{bmatrix}}$$

Multiplying a matrix by a vector

$m \times n \quad n \times 1 \quad m \times 1$

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n \end{bmatrix}$$

a scal

each entry of b is a row-column vector multiplication!

Mathematically:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j$$

Does this explain why n must be same for $Ax=b$? Yes!

This could be re-written as a sum of vectors: (weighted sum of the columns of A:

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \dots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

New "column" Interpretation

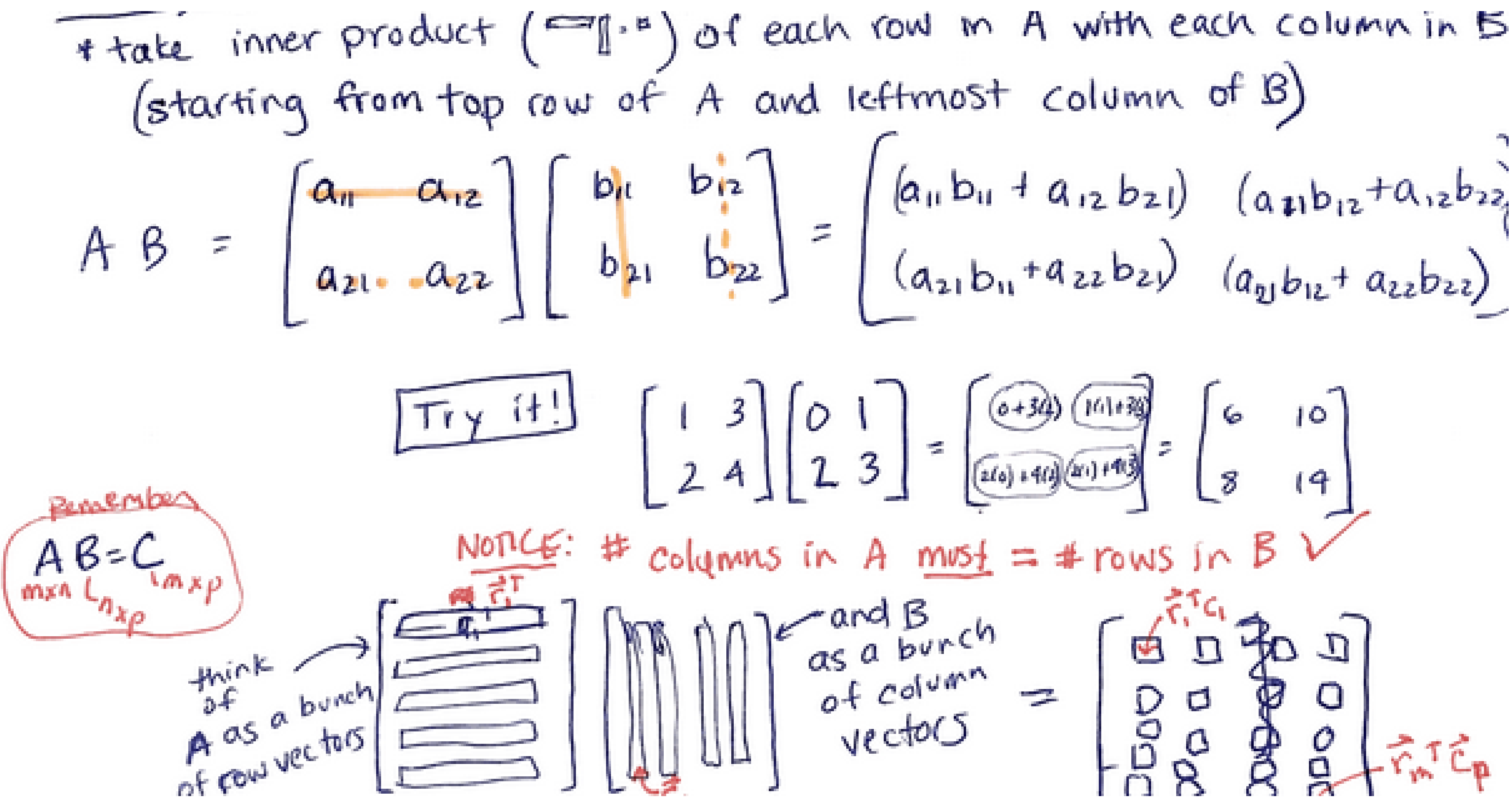
$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

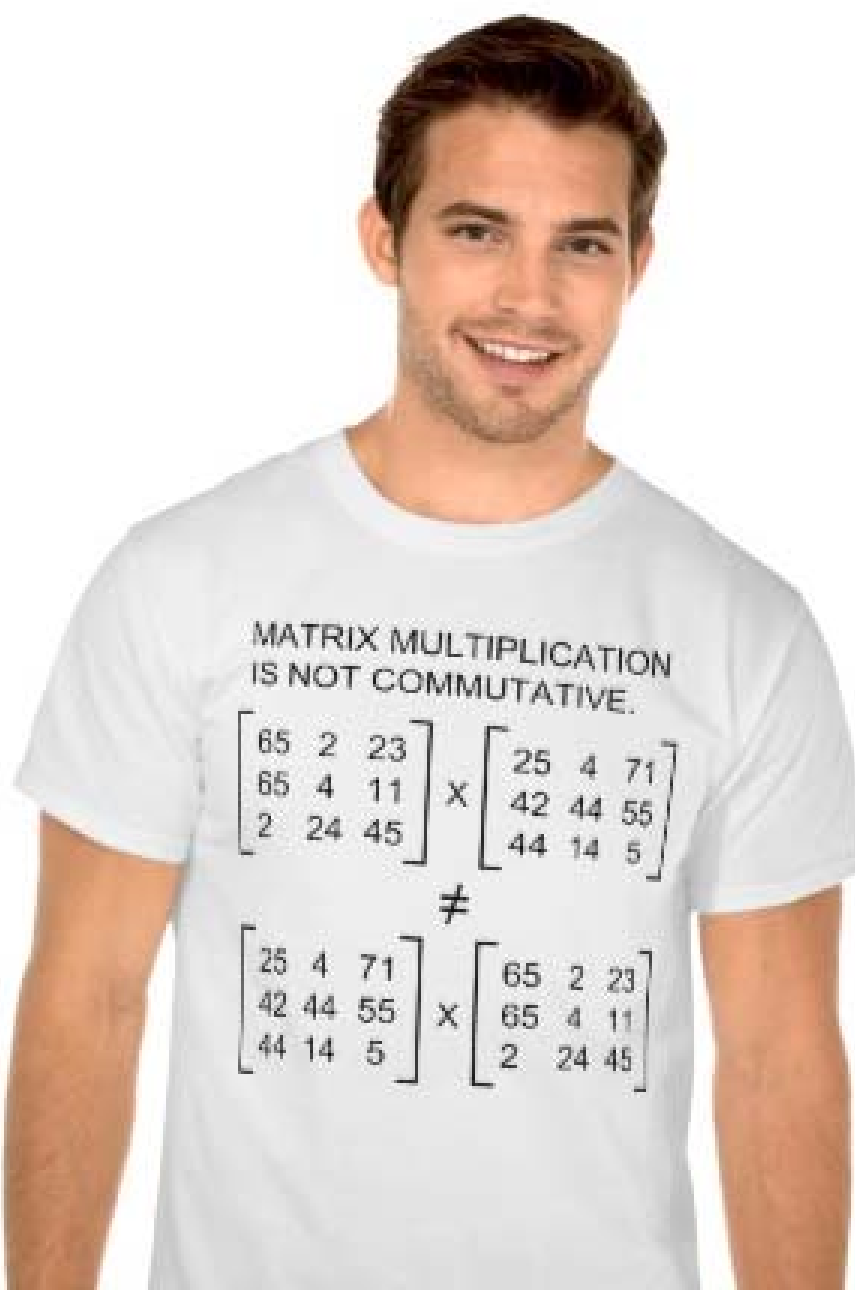
Example:

$$\begin{bmatrix} -1 & 3 \\ 3 & 2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$$

unit vector "selects" one column of matrix.

Multiplying a matrix by a matrix




MATRIX MULTIPLICATION
IS NOT COMMUTATIVE.
$\begin{bmatrix} 65 & 2 & 23 \\ 65 & 4 & 11 \\ 2 & 24 & 45 \end{bmatrix} \times \begin{bmatrix} 25 & 4 & 71 \\ 42 & 44 & 55 \\ 44 & 14 & 5 \end{bmatrix} \neq \begin{bmatrix} 25 & 4 & 71 \\ 42 & 44 & 55 \\ 44 & 14 & 5 \end{bmatrix} \times \begin{bmatrix} 65 & 2 & 23 \\ 65 & 4 & 11 \\ 2 & 24 & 45 \end{bmatrix}$

Last time: Linear systems of equations

$$\begin{aligned} ax_1 + ax_2 \qquad\qquad &= b_1 \\ ax_3 + ax_4 &= b_2 \\ ax_1 \qquad + ax_3 \qquad &= b_3 \\ ax_2 \qquad + ax_4 &= b_4 \end{aligned}$$

Can also be represented as:

$$\left[\begin{array}{cccc|c} a & a & 0 & 0 & b_1 \\ 0 & 0 & a & a & b_2 \\ a & 0 & a & 0 & b_3 \\ 0 & a & 0 & a & b_4 \end{array}\right]$$

$=$

Or:

$$\begin{bmatrix} a & a & 0 & 0 \\ 0 & 0 & a & a \\ a & 0 & a & 0 \\ 0 & a & 0 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Or:

$$Ax = b$$

Recall Last Lecture:

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}} \underset{n \times 1}{\overset{\vec{x} \text{ variables}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}} = \underset{m \times 1}{\overset{\vec{b} \text{ measurements}}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}}$$

$m \times n$: # equations (measurements); # unknowns (variables)

To solve, use Gaussian Elimination → scale and add different rows (equations)
→ swap order
Try to make A upper triangular

Good question: $\boxed{\text{What if } m>n \text{ or } n>m \text{ (A is not square)?}}$

- $m>n$: more equations than unknowns
  - ↳ some will be ~~redundant~~ redundant (useless)
  - ↳ does not mean it's solvable!
- $n>m$: not enough measurements!

For now, assume $m=n$ (A is square)

Possible cases:

all zeros on LHS was Red Flag!

① Row of zeros. $0\ 0\ 0 \ldots \mid b \neq 0$ ⇒ No Solution!

② Row of ALL zeros $0\ 0\ 0 \ldots \mid 0$ ⇒ Infinitely many solutions (if $m>n$, extra zero rows don't count)

③ If i^{th} row is $0\ 0\ 0 \ldots a_{ij} * * \mid *$ ⇒ UNIQUE solution

a_{ij}: not zero; $*$: other stuff (doesn't matter what)

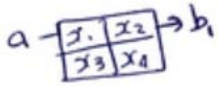
Note: As soon as you run into an inconsistent Row $0\ 0\ 0 \ldots 0 \mid b \neq 0$ ANYWHERE, give up! There is No solution!

we never used b to choose what to multiply rows by before adding to...

KEY POINT: we don't need to know $\vec{b}$ to know whether there is a unique sol'n! Only depends on SYSTEM, not Measurements → This is why we separate A and x

Example: Tomography: choice of how to take measurements mattered (~~linear combinati~~ what variables to add/scale)

↳ this is part of A



$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & A & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$A \quad \hat{x} = \vec{b}$$

What actual measurements were didn't matter

What if $b_1 \neq ax_1 + ax_2$? → meas. can only make set of equations INCONSISTENT

↳ this would mean model is wrong (doesn't capture physics) or meas. had error.

↳ Not the SYSTEM'S fault!

"Row-oriented perspective" → e.g. Gaussian Elimination looks for Rows of all zeros to find red flags for non-solvable problems

What do rows represent?

How much that variable affects that measurement.

Could I instead manipulate columns ~~to look for all-zero columns~~?

Let's think a bit more about this:

j^{th} column of A is $\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

What do these coefficients represent?

influence of variable x_j in different measurements (sensitivity)

They are all about one unknown (it's scaled value for each meas.)