$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

EE16A

Designing Information Devices and Systems I

Last time: Multiplying matrices/vectors

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ b_{21} & & b_{2p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1p} \\ c_{21} & & c_{2p} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mp} \end{bmatrix}$$

$$m \times n \qquad n \times p \qquad m \times p$$

Must be same!

Systems of equations $A\vec{x} = \vec{b}$

$$\begin{array}{cccc} \text{'system' matrix} & \text{solve for me!} & & \text{measurements} \\ \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} & = & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\ m \times n & n \times 1 & & m \times 1 \end{array}$$

equations/ measurements

unknowns

Last time: Row view

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n = b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots a_{mn}x_n = b_m \end{array}$$

What do rows represent? How much the variables affect a particular measurement.

Last time: Column view

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots + x_n \vec{a}_n}_{\text{Linear Combination of } \vec{a} \text{ vectors weighted by the unknowns!}} = \vec{b}$$

What do columns represent? How much a particular variable affects all measurements (sensitivity to that variable).

What if one a-vector is zeros? Then that variable not measured (could be anything)! No unique solution

This time: linear combinations

→ <u>Any</u> scaling or addition of vectors

Definition:

indicates a set

of vectors

where all elements are real

each vector has dimension N

For a set of vectors $\left\{\vec{a}_1, \vec{a}_2, \ldots \vec{a}_M\right\} \in \mathbb{R}^N$ where $\left\{\alpha_1, \alpha_2, \ldots \alpha_M\right\} \in \mathbb{R}$

some coefficients

are scalars

is defined as

then $\vec{w} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \ldots + \alpha_M \vec{a}_M$ is called a **<u>linear combination</u>** of the a-vectors

some new vector

Example: what are some linear combinations of $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$?

ex: $5\vec{x}_1 + 2\vec{x}_2$

$-6\vec{x}_1 - \vec{x}_2$

$0\vec{x}_1 + 5\vec{x}_2$

$\boxed{\text{What does it mean to solve } A\vec{x} = \vec{b}\text{?}}$ (w.r.t. linear combinations)

with respect to

→ Find the linear combination weights (x's) for columns of A that result in $\vec{b}$.

recall Column view:

$$A\vec{x} = \vec{b} \Rightarrow x_1 \begin{bmatrix} | \\ \vec{a}_1 \\ | \end{bmatrix} + x_2 \begin{bmatrix} | \\ \vec{a}_2 \\ | \end{bmatrix} + \dots x_n \begin{bmatrix} | \\ \vec{a}_n \\ | \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

notation just shows it's a col vector

i.e. for what setting of the unknowns do we get the measurements in $\vec{b}$?

Let's look only at A ⇒ the SYSTEM (before having taken measurements)

↳ how measurements were taken.

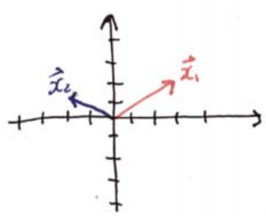
NEW JARGON ALERT! The $\boxed{\text{SPAN}}$ of a set of vectors is the set of ALL POSSIBLE linear combos of the vectors in the set.

$$\text{span}\underbrace{\left\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_M\right\}} \triangleq \left\{ \sum_{m=1}^{M} \alpha_m \vec{a}_m \;\middle|\; \underbrace{\alpha_1, \alpha_2 \dots \alpha_M \in \mathbb{R}}_{\text{coeffs. are scalar real}} \right\}$$

the sum over $m=1,2,\dots M$

"The span of $\boxed{\text{a set of vectors}}$ $\boxed{\text{is defined as}}$ $\boxed{\text{the set of all possible linear combinations of } \vec{a}_1, \vec{a}_2, \dots \vec{a}_n}$"

Try it! $\boxed{\text{What is the span of } \vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}\text{?}}$

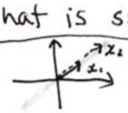


Entire 2D plane!

Because we can choose α_1, α_2 such that we can reach any point on Plane by $\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2$.

$\boxed{\text{Is this always true?}}$ No

$\boxed{\text{What is span of } \vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}\text{?}}$ A line!



Cannot choose α_1, α_2 to escape line!

These vectors are LINEARLY DEPENDENT

What is span of $\vec{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$? entire 3D space!

What is span of $\vec{0}$? $\vec{0}$ This is the one vector you can always reach (always in span).

New Jargon Alert!

If $\underbrace{\exists}_{\text{there exists}} \overset{\text{a vector}}{\vec{x}}$ $\underbrace{\text{s.t.}}_{\text{such that}}$ $A\vec{x} = \vec{b}$ then we say $\vec{b} \underset{\text{are in}}{\in} \overset{\text{all the places you can get to}}{\text{span(columns of A)}}$

What if $\vec{b}$ is not in span(cols of A)?

If measurements $\vec{b}$ not reachable by system A, then measurements are wrong (inconsistent) ⇒ No Solution

~~Now, let's reinterpret Gauss. Elimination:~~

Example:

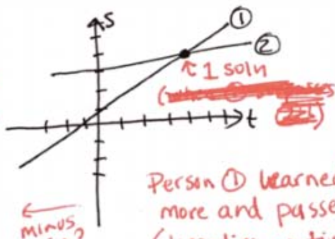
UNIQUE SOL'N

$t - s = -1$ ① Person

$0.1t - s = -4$ ② Person

t = time

s = skills



Person ① learned more and passed ②! (depending on time length of semester)

$$\begin{bmatrix}1 & -1\\0.1 & -1\end{bmatrix}\begin{bmatrix}t\\s\end{bmatrix} = \begin{bmatrix}1\\-4\end{bmatrix}$$

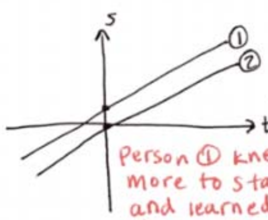
initial skills

*Takeaway: slope (hard work) can trump y-intercept, and it's the only thing you have control of! may take time.

NO SOL'N

$t - s = -1$ ①

$t - s = 0$ ②



Person ① knew more to start and learned as fast as ②.

$$\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}\begin{bmatrix}t\\s\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

parallel lines have same slope

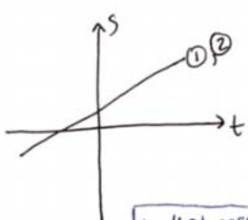
What does A say about line? LHS of ① and ② tells you SLOPE!

what does intercept depend on? $\vec{b}$ and slope

INFINITE SOL'NS

$t - s = -1$ ①

$2t - 2s = -2$



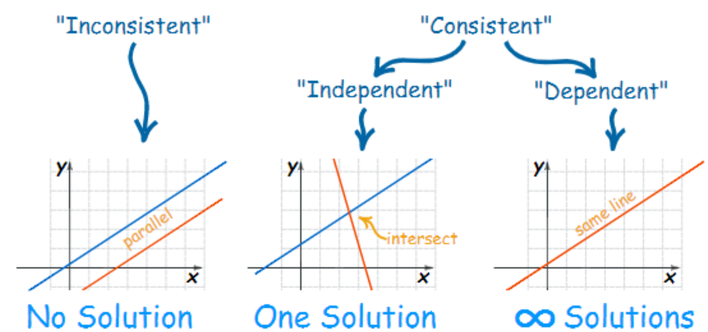
$$\begin{bmatrix}1 & -1\\2 & -2\end{bmatrix}\begin{bmatrix}t\\s\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

same lines have same slope + intercept.

~~Does it extend to 3D? Yes, planes intersect~~

*when rows multiples of each other → same slope

In that case, columns also multiples!



So if rows or colums are multiples of each other, then ~~of~~ they are REDUNDANT ~~REDUNDANT~~

$\boxed{\text{What does redundant mean?}}$ ⇒ LINEAR DEPENDENCE

Linear dependence: (2 definitions)

$\boxed{\text{Definition 1}}$

A set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3 \dots \vec{a}_M\}$ is linearly dependent if

$$\underbrace{\vec{a}_j = \sum_{m \neq j} \alpha_m \vec{a}_m \text{ for some } \{\alpha_m \in \mathbb{R}\}_{m \neq j} \text{ and some } 1 \leq j \leq M}_{\text{one vector can be written as linear combo. of others}}$$

Example: $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

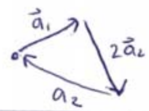
Can any vectors in set be written as linear combo. of others?

$$\vec{a}_3 = \vec{a}_1 + 2\vec{a}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \boxed{= \vec{a}_3}$$

so set is LINEARLY DEPENDANT

another way to say it: $\vec{0} = \vec{a}_1 + 2\vec{a}_2 - \vec{a}_3$

can i get back to where i started?



*fits with row-view from Gauss. Elim., but this is col-view

This leads to:

$\boxed{\text{Definition 2}}$

A set of vectors $\{\vec{a}_1, \vec{a}_2, \dots \vec{a}_m\}$ all in $\mathbb{R}^N$ is linearly dependent if there are $\alpha_1, \alpha_2, \dots \alpha_M \in \mathbb{R}$

scalar coeffs.

$$\text{s.t. } \sum_{m=1}^{M} \alpha_m \vec{a}_m = \vec{0}$$

such that

linear combo.s of all vectors $= \vec{0}$

BUT need to exclude trivial case of all $\alpha_i = 0$, so not all $\alpha_i = 0$

$\boxed{\text{Are these two def'ns equivalent?}}$ Yes!

$\boxed{\text{What is definition of linear independence?}}$ ≡ NOT linearly dependent

$\boxed{\text{T or F: Can } \vec{0} \text{ be lin. ind. set of vectors?}}$ No eg: $\{\vec{0}, \vec{a}_1, \vec{a}_2 \dots \vec{a}_M\}$

$\hookrightarrow \vec{0} = 0\vec{a}_1 + 0\vec{a}_2 + \dots 0\vec{a}_3$

(Def'n ① doesn't require not all α zero!)

Example: Is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ lin. dependent? No

Properties of $A\vec{x}=\vec{b}$ systems of equations:

① $A\vec{x}=\vec{b}$ has a solution iff $\vec{b} \in \text{span}(\text{cols of } A)$

if and only if — means — are in span of columns of A

② $A\vec{x}$ is a linear combination of the columns of A (column-view)

③ unique solution $\equiv$ unique $\vec{x}$ that satisfies $A\vec{x}=\vec{b}$

is defined as

NEW and Important ④ Every $\vec{b} \in \text{span}(\text{cols of } A)$ is uniquely spanned iff the colums of A are Linearly Independent

$\boxed{\text{~~What does lin. independent mean?~~}}$ ~~Not lin. dependent.~~

Possible cases:

$\vec{b} \notin \text{span}(\text{cols of } A) \longrightarrow$ No solution

not in

$\vec{b} \in \text{span}(\text{cols of } A)$
- → if (cols A) linearly independent → 1 solution
- → if (cols A) linearly dependent → ∞ solutions

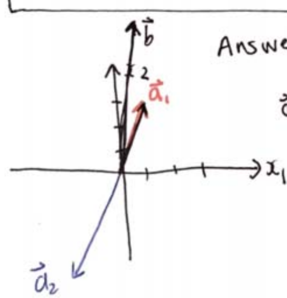
Example:

old example (had no sol'n)

$$\begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array}$$

$$\begin{bmatrix} 1 & -3 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$$

$$\boxed{\text{Can I write } \begin{bmatrix} 1 \\ 9 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ -9 \end{bmatrix} ?}$$



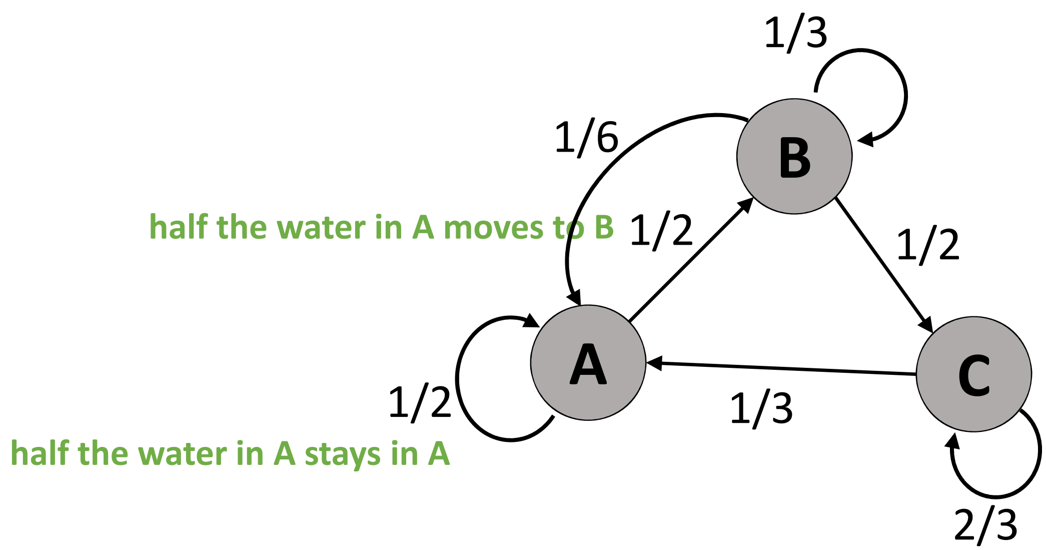
Answer: No

$\vec{a}_1, \vec{a}_2$ Same line! Linearly Dependent

Cannot drive different amounts along direction $\vec{a}_1$ and $\vec{a}_2$ to get to $\vec{b}$

Graph Representation

New example problem: Reservoirs and Pumps



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

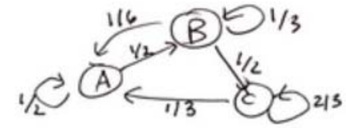
Where the water moves and what fraction is represented by arrows.

Edge weights **Edges**

"directed" graph because arrows have a direction

Where does the rest of the water in A go? Need to label that too…

Can you tell me how much water in each after pumps start? Need to know initial amounts



Reset Pumps Lec 02

Say initial water levels (before pumps start) are $\vec{s} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$ ← water in reservoir (A), ← water in reservoir (B)

$\boxed{\text{How much water in reservoirs (nodes) after pumps run?}}$

We are looking for solution $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ← water in res. (A) after pumps, ← water in res. (B) after pumps

Read off graph:

$$\vec{y} = \begin{bmatrix} \frac{1}{2} s_1 + \frac{1}{6} s_2 + \frac{1}{3} s_3 \\ \frac{1}{2} s_1 + \frac{1}{3} s_2 + 0 s_3 \\ 0 s_1 + \frac{1}{2} s_2 + \frac{2}{3} s_3 \end{bmatrix} = s_1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} \frac{1}{6} \\ \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} + s_3 \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$$

s_1 coeffs are OUTFLOW for Res (A) — s_2 coeffs — s_3 coeffs.

column view → OUTFLOWS

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

inflow for Res (A)!

Row view → Inflows

$$\vec{y} = \underbrace{P}_{\text{matrix}} \vec{s}$$

← system is applied to state

This is a matrix-vector multiplication!

useful representation tool.

$\boxed{\text{What else could nodes and edges represent?}}$

e.g. - people & flow/traffic
- money & purchases
- etc.

$$P_{matrix} = \begin{bmatrix} P_{A\to A} & P_{B\to A} & P_{C\to A} \\ P_{A\to B} & P_{A\to B} & P_{C\to B} \\ P_{A\to C} & P_{B\to C} & P_{C\to C} \end{bmatrix}$$

$\boxed{\text{Does the total outflow have to} = 1\text{?}}$

No, but then water not conserved! if cols total = 1

There is a sink (leak) e.g. $P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ → maybe it evaporated?

Example: If we start with $\vec{s} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}$, what are final water levels?

$$\vec{y} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1+0 \\ 1+2+0 \\ 0+3+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

x2

What if I start with 2x as much?

First, think intuitively...

$$\vec{s}\,' = \begin{bmatrix} 4 \\ 12 \\ 0 \end{bmatrix} \rightarrow \vec{y}\,' = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/2 & 1/3 & 0 \\ 0 & 1/2 & 2/3 \end{bmatrix} \begin{bmatrix} 4 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 6 \end{bmatrix}$$

output is 2x as much! Makes sense cause they are fractions in P matrix.

Observations:

① if $\underset{\text{input}}{\vec{s}} \xrightarrow{P} \underset{\text{output}}{\vec{y}}$, then $\alpha\vec{s} \xrightarrow{P} \alpha\vec{y}$ — input/output relationship is scaled by same

run pumps once

consequence of LINEARITY

② if $\vec{s} \xrightarrow{P} \vec{y}$ and $\vec{v} \xrightarrow{P} \vec{z}$ $\Big\}$ $\vec{s} + \vec{v} \longrightarrow \vec{y} + \vec{z}$ — can add inputs OR outputs

③ We can think of pumps as transformation T

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ tells me water levels in 3 ~~pumps~~ res.'s after pumps

treat like a function: we know
$T(\vec{s}) = \vec{y}$
$T(\vec{v}) = \vec{z}$
$T(\alpha\vec{s}) = \alpha T(\vec{s})$
$T(\vec{s}) + T(\vec{v}) = T(\vec{s} + \vec{v})$

What if I run pumps twice?

Take output from first run & use as input from second.

ex. $\vec{s} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} \rightarrow \vec{y} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ for pumps run once, so new $\vec{y}\,' = \begin{bmatrix} P \end{bmatrix} \underset{\vec{s}\,'}{\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}}$

$$= \begin{bmatrix} 1+0.5+1 \\ 1.5+1+0 \\ 0+1.5+2 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \\ 3.5 \end{bmatrix}$$

Is total water level conserved?

Yes