Inner Products

Definition of Inner Product: The (Euclidean) inner product between two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is defined as:

$$\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &\equiv \vec{x}^T \vec{y} && (1)\\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} && (2)\\
&= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n && (3)\\
&= \sum_{i=1}^{n} x_i y_i && (4)
\end{aligned}$$

In physics, inner products are often called **dot products** (denoted by $\vec{x} \cdot \vec{y}$) and can be used to calculate values such as work. In this class we will use the notation $\langle \vec{x}, \vec{y} \rangle$ for the inner product.

Example 21.1 (Inner product of two vectors): Compute the inner product of the two vectors $\begin{bmatrix} -1 & 3.5 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$.

$$\begin{aligned}
\left\langle \begin{bmatrix} -1 \\ 3.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\rangle &= \begin{bmatrix} -1 & 3.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} && (5)\\
&= -1 \times 1 + 3.5 \times 0 + 0 \times 2 && (6)\\
&= -1 + 0 + 0 = -1. && (7)
\end{aligned}$$

Properties of Inner Products

Symmetry: The inner product is a *symmetric* operation; that is, it remains the same even if we switch its arguments.
Proof:

$$\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &= x_1 y_1 + \cdots + x_n y_n \\
&= y_1 x_1 + \cdots + y_n x_n \qquad (8)\\
&= \langle \vec{y}, \vec{x} \rangle
\end{aligned}$$

Homogeneity: If we scale $\vec{x}$ by a real number c, we can show that scaling any vector in the inner product by any real constant c will scale the inner product by the same constant.
Proof:

$$\begin{aligned}\langle c\vec{x}, \vec{y}\rangle &= (cx_1)y_1 + \cdots + (cx_n)y_n \\ &= c(x_1y_1) + \cdots + c(x_ny_n) \\ &= c\,\langle \vec{x}, \vec{y}\rangle\end{aligned} \tag{9}$$

By a similar argument (exercise: prove it yourself),

$$\langle \vec{x}, c\vec{y}\rangle = c\,\langle \vec{x}, \vec{y}\rangle \tag{10}$$

Additivity: What happens when we take the inner product between a sum of two vectors and another vector? We can write:

$$\begin{aligned}\langle \vec{x}+\vec{y}, \vec{z}\rangle &= (x_1+y_1)z_1 + \cdots + (x_n+y_n)z_n \\ &= x_1z_1 + y_1z_1 + \cdots + x_nz_n + y_nz_n \\ &= x_1z_1 + \cdots + x_nz_n + y_1z_1 + \cdots + y_nz_n \\ &= \langle \vec{x}, \vec{z}\rangle + \langle \vec{y}, \vec{z}\rangle\end{aligned} \tag{11}$$

By a similar argument:

$$\langle \vec{x}, \vec{y}+\vec{z}\rangle = \langle \vec{x}, \vec{y}\rangle + \langle \vec{x}, \vec{z}\rangle \tag{12}$$

Homogeneity together with additivity gives rise to a property called *bilinearity*. This means that the inner product is linear in each argument. These properties are also very useful in proving properties other about inner products.

Orthogonal Vectors

Two vectors $\vec{x}$, $\vec{y}$ are said to be **orthogonal** if their inner product is zero, i.e. $\langle \vec{x}, \vec{y}\rangle = 0$. In 2D and 3D coordinate spaces, perpendicular and orthogonal mean the same thing; in higher-dimension spaces, it is harder to visualize vectors being perpendicular, so the term orthogonal comes in handy to abstract away the need to visualize.

Here's an example, in $\mathbb{R}^3$. Let's say $\vec{x} = \begin{bmatrix}1\\1\\3\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}4\\-1\\-1\end{bmatrix}$. Then, $\langle \vec{x}, \vec{y}\rangle = (1)(4) + (1)(-1) + (3)(1) = 0$. Thus $\vec{x}$ and $\vec{y}$ are orthogonal.

Note that the standard unit vectors are always orthogonal to each other.

Special Vector Operations

The inner product is a basic building block for many operations. Here are some useful operations you can perform with the inner product calculation using different vectors as inputs.

These things are often important in computer programming contexts because computers (and programming languages) are often optimized to be able to do vector operations like inner products very fast. So, seeing an inner-product way of representing something can often speed up calculations considerably.

Sum of Components

$$\left\langle \begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T, \begin{bmatrix}x_0 & x_1 & \cdots & x_n\end{bmatrix}^T \right\rangle = x_0 + x_1 + \ldots + x_n \tag{13}$$

Average

$$\left\langle \begin{bmatrix}\frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n}\end{bmatrix}^T, \begin{bmatrix}x_0 & x_1 & \cdots & x_n\end{bmatrix}^T \right\rangle = \frac{x_0 + x_1 + \ldots + x_n}{n} \tag{14}$$

Sum of Squares

$$\left\langle \begin{bmatrix}x_0 & x_1 & \cdots & x_n\end{bmatrix}^T, \begin{bmatrix}x_0 & x_1 & \cdots & x_n\end{bmatrix}^T \right\rangle = x_0^2 + x_1^2 + \ldots + x_n^2 \tag{15}$$

Selective Sum Here, the first vector has values of 0's and 1's, where 1's correspond to the values of the second vector that are to be used in the sum. This, for example, can take the place of a for loop checking every element of a vector.

$$\left\langle \begin{bmatrix}0 & 0 & 1 & 0 & 1 & \cdots & 0 & 1\end{bmatrix}^T, \begin{bmatrix}x_0 & x_1 & \cdots & x_n & \end{bmatrix}^T \right\rangle = x_2 + x_4 + \ldots + x_n \tag{16}$$

Introduction to Norms

The **Euclidean Norm** of a vector is defined as:

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\langle \vec{x}, \vec{x} \rangle} \tag{17}$$

Why is the norm important? The 2-norm of a vector is also the magnitude of the vector (or length of the arrow). This corresponds to the usual notion of distance in $\mathbb{R}^2$ or $\mathbb{R}^3$. It is interesting to note that the set of points with equal Euclidean norm is a circle in $\mathbb{R}^2$ or a sphere in $\mathbb{R}^3$.

You may have noticed that the subscript 2 in the definition of the norm given above. The subscript differentiates the Euclidean norm (or 2-norm) from other useful norms. In general, the p-norm is defined as:

$$\|\vec{x}\|_p = \left(|x_1|^p + |x_2|^p + \ldots + |x_n|^p\right)^{1/p} \tag{18}$$

These other norms might feel esoteric but turn out to be useful in many engineering settings. Follow-on courses like EE 127 and EE 123 will show you how these can be useful in applications like machine learning.

If no subscript is used, you can assume the 2-norm.

Properties of Norms

$$\|\alpha x\| = |\alpha| \, \|x\| \tag{19}$$

$$\|x + y\| \leq \|x\| + \|y\| \text{, known as the "triangle inequality"} \tag{20}$$

$$\|x\| \geq 0 \tag{21}$$

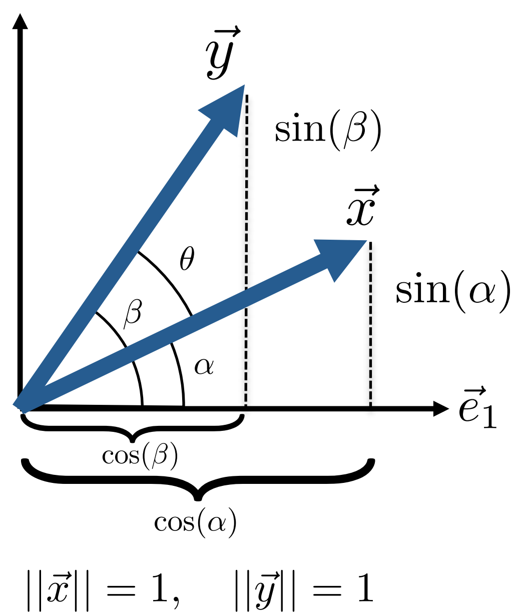
$$\|x\| = 0 \text{ only if } x = 0 \tag{22}$$

Interpretations of the Inner Product

Now that we have defined the inner product, it's time to get an intuition about what an inner product is.

First we take the unit vector $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and a general unit vector in $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$. $\vec{x}$ is a unit vector because $\|\vec{x}\| = \sqrt{\cos^2\alpha + \sin^2\alpha} = 1$.

When we draw the vectors, the angle between them is α. We can calculate $\langle \vec{e}_1, \vec{x} \rangle = 1 \times \cos\alpha + 0 \times \sin\alpha = \cos\alpha$. With this we might guess that the inner product of two unit vectors is the cosine of the angle between them. Let's try to prove this for two general unit vectors in $\mathbb{R}^2$, $\vec{x} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$. (All unit vectors can be represented in this form).



Using the trigonometric identity $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$, we find:

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= \cos(\alpha - \beta) \\ &= \cos\theta \end{aligned} \tag{23}$$

Again, that is the cosine of the angle between these two vectors!

What about any two general vectors $\vec{x}$ and $\vec{y}$? We can first convert them into the unit vectors $\vec{x}/\|x\|$ and $\vec{y}/\|y\|$ by dividing each vector by its norm. Then we can use (23) to get:

$$\left\langle \frac{\vec{x}}{\|x\|}, \frac{\vec{y}}{\|y\|} \right\rangle = \cos\theta \tag{24}$$

We can then use the homogeneity property (9) since both norms are scalars:

$$\begin{aligned} \frac{\langle \vec{x}, \vec{y} \rangle}{\|x\| \, \|y\|} &= \cos\theta \\ \langle \vec{x}, \vec{y} \rangle &= \|x\| \, \|y\| \cos\theta \end{aligned} \tag{25}$$

Now we have a geometric interpretation of the inner product: the inner product of vectors $\vec{x}$ and $\vec{y}$ is their lengths multiplied by the angle between them. One remarkable observation is that the inner product *does not depend on the coordinate system the vectors are in*, it only depends on the relative angle between these vectors and their length. This is the reason it is very useful in physics, where the physical laws do not depend on the coordinate system used to measure them. This is also the reason why this property holds in higher dimensions as well. For any two vectors we can look at the plane passing through them and the angle between them is the angle θ measured in the plane.

The Cauchy-Schwarz Inequality

The Cauchy-Schwarz inequality relates the inner product of two vectors to their length. The Cauchy-Schwarz inequality states:

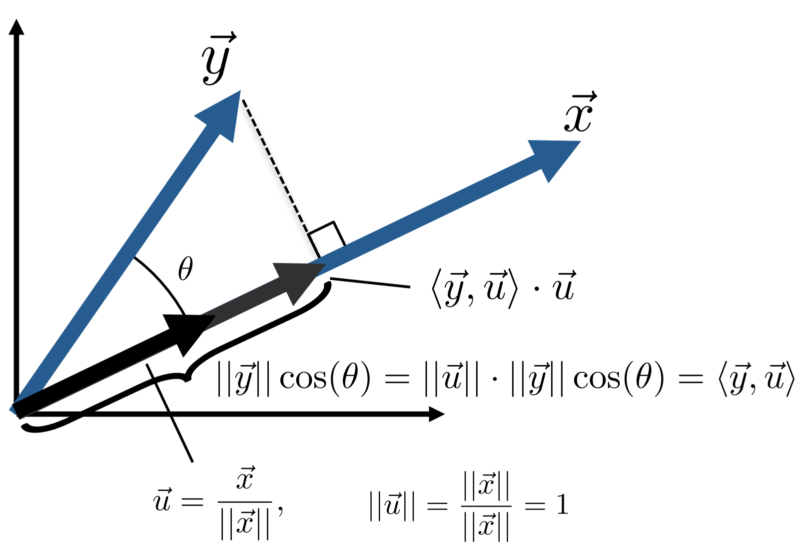
$$|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\| \tag{26}$$

We can prove this by recognizing that $|\cos\theta| \leq 1$ for all real θ. Thus:

$$\begin{aligned} |\langle \vec{x}, \vec{y} \rangle| &= |\, \|\vec{x}\| \, \|\vec{y}\| \cos\theta| \\ &= \|\vec{x}\| \, \|\vec{y}\| \cdot |\cos\theta| \\ &\leq \|\vec{x}\| \, \|\vec{y}\| \end{aligned} \tag{27}$$

Projections

Knowing that the inner product of two vectors in $\mathbb{R}^n$ is the product of their lengths and the angle between them, we can write the *projection* of one vector onto another using the inner product.



The projection of $\vec{y}$ onto $\vec{x}$ is the component of $\vec{y}$ lying in the direction of $\vec{x}$. First let's find the length of this component. By geometry, this is $\|\vec{y}\|\cos\theta$, where θ is the angle between $\vec{y}$ and $\vec{x}$. We can write it as $\langle\vec{y},\vec{u}\rangle$, where $\vec{u} = \vec{x}/\|\vec{x}\|$ is the unit vector in the direction of $\vec{x}$. Now we know the length of the projection, all we need is a direction. This is just $\vec{u}$, so the projection of $\vec{x}$ onto $\vec{y}$ is given by:

$$\langle\vec{y},\vec{u}\rangle\,\vec{u} = \left\langle \vec{y}, \frac{\vec{x}}{\|\vec{x}\|} \right\rangle \frac{\vec{x}}{\|\vec{x}\|} \tag{28}$$