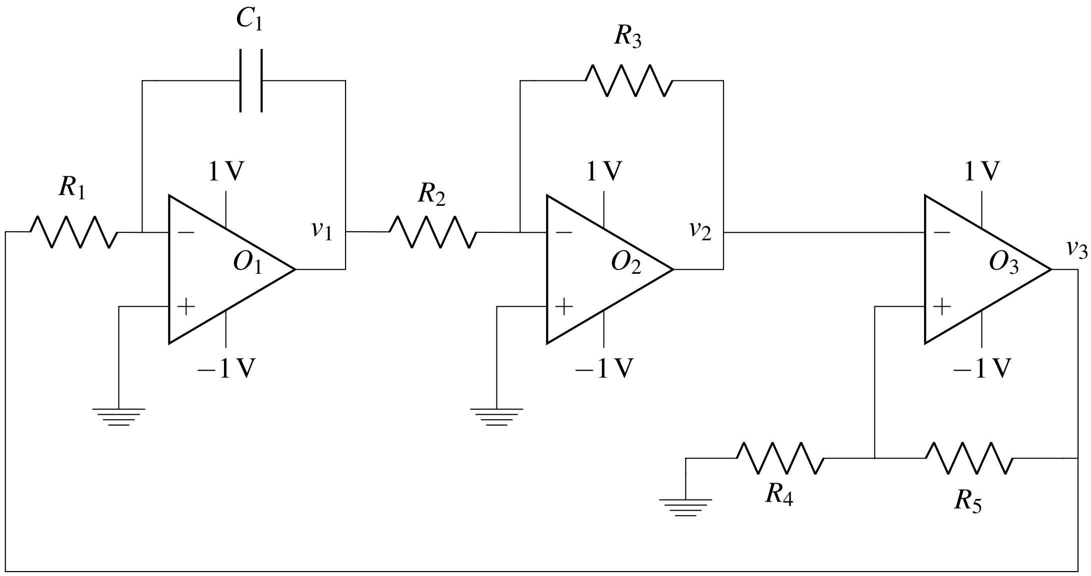
EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 11A

1. Timer Circuit

In this problem, we will walk through the timer circuit, shown below, similar to the one seen in lecture. The circuit is shown below. All resistors have a resistance of $1\,\mathrm{k}\Omega$ and $C_1 = 1\,\mu\mathrm{F}$.



(a) Find the current through the capacitor C_1 in terms of the voltage V_3 and the resistor R_1.

Answer:

For an op-amp, no current flows into the input terminals. Therefore, all the current through R_1 must flow through C_1. Applying the Golden Rules, we know that $v_+ = v_- = 0\,\mathrm{V}$.

$$i_{R_1} = i_{C_1} = \frac{v_3}{R_1}$$

(b) Suppose that at time $t = 0$, C_1 is uncharged. Find the voltage v_1 in terms of t, v_3, and R_1. What is the maximum $|v_1|$ could be?

Answer:

Recall the voltage across a capacitor is related to the charge on the capacitor, that is $Q = CV$. Current is related to charge with the equaion $I = \frac{dQ}{dt}$.

$$v_{C_1} = \frac{Q}{C_1} = \frac{1}{C_1} It = \frac{v_3}{R_1 C_1} t = \frac{v_3}{1\,\mathrm{ms}} t$$

Note that a ΩF is a second. Because the current is flowing into the capacitor, as the voltage across the capacitor increases, the output voltage decreases.

$$v_1 = -v_{C_1} = -\frac{v_3}{1\,\mathrm{ms}} t$$

The maximum or minimum for v_1 is the top or bottom supply rail, so either $+1\,\text{V}$ or $-1\,\text{V}$. Therefore, the maximum $|v_1| = 1\,\text{V}$.

(c) How is v_2 related to v_1? What is the voltage v_2?

Answer:

O_2 is an inverting amplifier. The output voltage v_2 is equal to $-v_1$.

$$v_2 = \frac{v_3}{1\,\text{ms}} t$$

O_3 is not connected in negative feedback. However, we can analyze its behavior by considering it to be a comparator. Let's independently analyze the circuit in the two possible outputs of the comparator, when $v_3 = 1\,\text{V}$ and when $v_3 = -1\,\text{V}$.

(d) Assume that the output of the comparator v_3 has railed to the top rail. With this value of v_3, what is v_2 as a function of time? What is the voltage at the positive input of O_3? At what time will the two inputs of the comparator be equal?

Answer:

With v_3 at the top rail, v_2 is $\frac{t}{1\,\text{ms}}\text{V}$. The voltage at the positive input of the opamp is $0.5\,\text{V}$ because of R_5 and R_4. Therefore, when $t = 0.5\,\text{ms}$, $v_2 = 0.5\,\text{V}$.

(e) Now assume that the reverse occurs, that is, the output of the comparator has railed to the bottom rail. Repeat part (d) with this value of v_3.

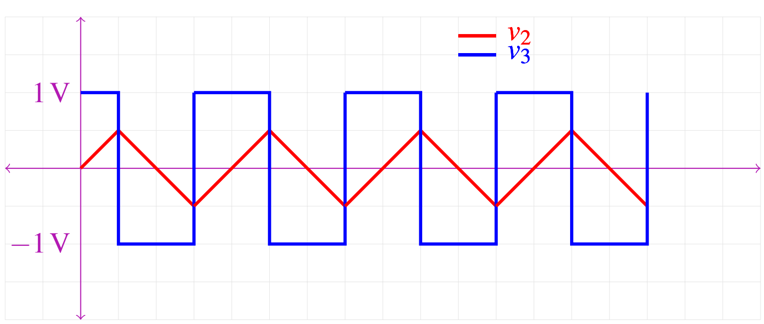
Answer:

With v_3 at the bottom rail, v_2 is $-\frac{t}{1\,\text{ms}}\text{V}$. Similar to part (d), the voltage at the positive input is $-0.5\,\text{V}$. Therefore, when $t = 0.5\,\text{ms}$, $v_2 = -0.5\,\text{V}$.

(f) What is v_3 as a function of time? Draw a graph of v_3 and v_2. Since the graph is periodic, find its period and frequency.

Answer:

Notice that in each of the above cases, once v_2 was equal to v_+, the output of the comparator would flip. This leads to a periodic function, where v_3 is either $+1\,\text{V}$ or $-1\,\text{V}$. The period of this function is $T = 2\,\text{ms}$. Notice that in each of the above cases we analyzed, we always assumed that the capacitor was initially uncharged. However, when v_3 switches, the capacitor will already have some charge built up on it, so it must first be drained. This is why the period is twice what we expect.



(g) Suppose that we changed the value of C_1 to be $2\,\mu\text{F}$? What is the new period? Suppose that we change R_5 to be $2\,\text{k}\Omega$. What is the new period? What if we change R_5 to be $0\,\Omega$? Will this circuit still operate?

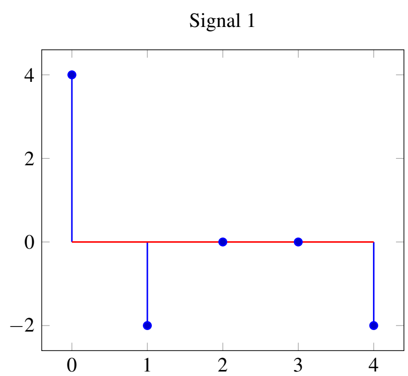
Answer:

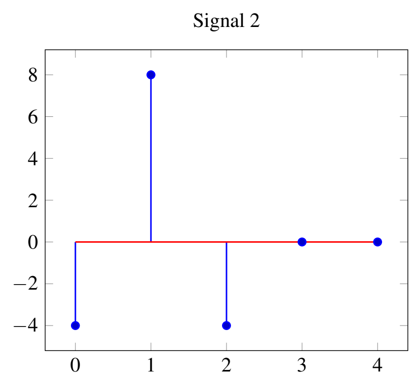
Notice above we got the constant 1 ms by multiplying R_1 and C_1 together. If we double C_1, the effective period would double because it would take longer to charge C_1 to the same voltage with the same current.

Changing R_5 affects the "flip" threshold because v_+ is at a different voltage. Increasing R_5 decreases the voltage at v_+, so we would expect the flip voltage to decrease. In fact, the new period is $\frac{4}{3}$ms.

The circuit would not operate if $R_5 = 0\,\Omega$. The inverting input needs to be able to go above and below the non-inverting input, which is not possible if the non-inverting input is constant at the rail.

2. Correlation You are given the following two signals:





(a) Assume the two signals are periodic with period 5. Find their linear cross correlation, that is find $\text{corr}(\vec{s}_1, \vec{s}_2)$.

Answer: For $\vec{x}$, $\vec{y}$ that are periodic with period N: $\text{corr}_N(\vec{x},\vec{y})[k] = \sum_{i=0}^{N-1} x[i]y[i-k]$

Since the signals are periodic they continue on to $+\infty$ and $-\infty$. We'll start by just performing the linear correlation over one period, and ignore the signals outside of this period. Shifting the signal back will bring the next period of the signal into our range of interest. Thus we calculate the linear cross-correlation assuming the signals are periodic as below:

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n]$	−4		8		−4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	−16	+	−16	+	0	+	0	+	0	$=-32$

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-1]$	0		−4		8		−4		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	8	+	0	+	0	+	0	$=8$

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-2]$	0		0		−4		8		−4	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	8	$=8$

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-3]$	−4		0		0		−4		8	
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$	−16	+	0	+	0	+	0	+	−16	$=-32$

$\vec{s}_1$	4		−2		0		0		−2	
$\vec{s}_2[n-4]$	8		−4		0		0		−4	
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$	32	+	8	+	0	+	0	+	8	$=48$

Let's continue to calculate the values of the inner product with more shifts.

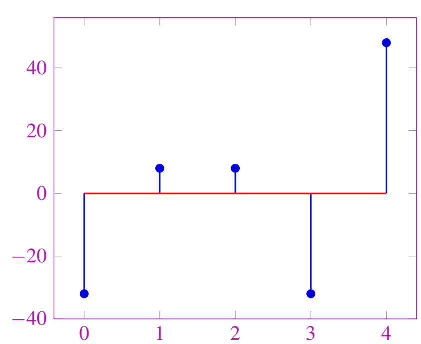
$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-5]$	-4		8		-4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n-5] \rangle$	-16	$+$	-16	$+$	0	$+$	0	$+$	0	$=-32$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-6]$	0		-4		8		-4		0	
$\langle \vec{s}_1, \vec{s}_2[n-6] \rangle$	0	$+$	8	$+$	0	$+$	0	$+$	0	$=8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-7]$	0		0		-4		8		-4	
$\langle \vec{s}_1, \vec{s}_2[n-7] \rangle$	0	$+$	0	$+$	0	$+$	0	$+$	8	$=8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-8]$	-4		0		0		-4		8	
$\langle \vec{s}_1, \vec{s}_2[n-8] \rangle$	-16	$+$	0	$+$	0	$+$	0	$+$	-16	$=-32$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-9]$	8		-4		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-9] \rangle$	32	$+$	8	$+$	0	$+$	0	$+$	8	$=48$



Notice that the pattern repeats, this leads to the definition of circular correlation, which we will explore in the later part.

(b) Sketch the linear cross-correlation of signal 1 with signal 2, that is find : $\text{corr}(\vec{s}_1, \vec{s}_2)$. Do not assume the signals are periodic.

Answer:

Represent signal 1 as the vector $\vec{s}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 4 & -2 & 0 & 0 & -2 \end{bmatrix}^T$, zero-padded so that we compute only the linear correlation. Similarly, represent signal 2 as the vector
$\vec{s}_2 = \begin{bmatrix} -4 & 8 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, where we once again zero pad the vector. Notice we zero pad the front of the vector $\vec{s}_2$ but the back of the vector $\vec{s}_1$.

The cross-correlation between two vectors is defined as follows:

$$\text{corr}(\vec{x}, \vec{y})[k] = \sum_{i=-\infty}^{\infty} \vec{x}[i]\vec{y}[i-k]$$

To compute the cross-correlation $\text{corr}(\vec{s}_1, \vec{s}_2)$, we shift the vector $\vec{s}_2$ and compute the inner product of the shifted $\vec{s}_2$ and the vector $\vec{s}_1$.

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+4]$	-4		8		-4		0		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+4] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+3]$	0		-4		8		-4		0		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+3] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	= 0

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+2]$	0		0		-4		8		-4		0		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+2] \rangle$	0	+	0	+	0	+	0	+	-16	+	0	+	0	+	0	+	0	= -16

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n+1]$	0		0		0		-4		8		-4		0		0		0	
$\langle \vec{s}_1, \vec{s}_2[n+1] \rangle$	0	+	0	+	0	+	0	+	32	+	-8	+	0	+	0	+	0	= 40

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n]$	0		0		0		0		-4		8		-4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	0	+	0	+	0	+	0	+	-16	+	-16	+	0	+	0	+	0	= -32

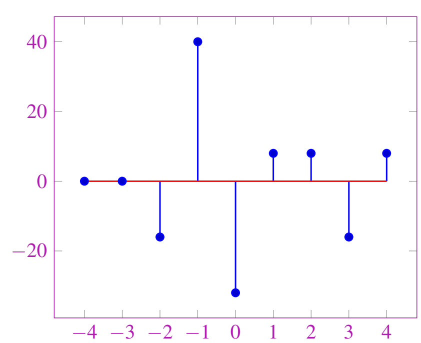
$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-1]$	0		0		0		0		0		-4		-8		-4		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	0	+	0	+	0	+	0	+	8	+	0	+	0	+	0	= 8

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-2]$	0		0		0		0		0		0		-4		8		-4	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	8	= 8

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-3]$	0		0		0		0		0		0		0		-4		8	
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	-16	= -16

$\vec{s}_1$	0		0		0		0		4		-2		0		0		-2	
$\vec{s}_2[n-4]$	0		0		0		0		0		0		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$	0	+	0	+	0	+	0	+	0	+	0	+	0	+	0	+	8	= 8

Non-periodic Cross-correlation of Signals 1 and 2



(c) Find the circular cross correlation of $\vec{s}_2$ with $\vec{s}_1$, that is find $\text{circcorr}(\vec{s}_1, \vec{s}_2)$

Answer:

Represent signal 1 as the vector $\vec{s}_1 = \begin{bmatrix}4 & -2 & 0 & 0 & -2\end{bmatrix}^T$. Similarly, represent signal 2 as the vector $\vec{s}_2 = \begin{bmatrix}-4 & 8 & -4 & 0 & 0\end{bmatrix}$

The cross-correlation between two vectors of length N is defined as follows:

$$\text{circcorr}(\vec{x}, \vec{y})[k] = \sum_{i=0}^{N-1} \vec{x}[i]\vec{y}[(i-k)_N]$$

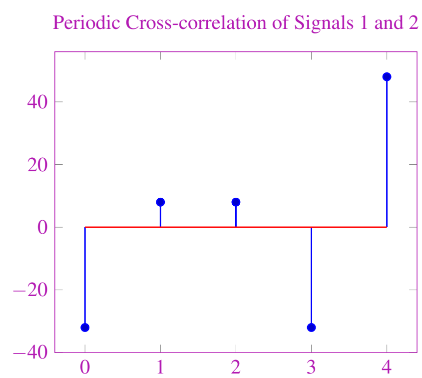
$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n]$	-4		8		-4		0		0	
$\langle \vec{s}_1, \vec{s}_2[n] \rangle$	-16	+	-16	+	0	+	0	+	0	$= -32$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-1]$	0		-4		8		-4		0	
$\langle \vec{s}_1, \vec{s}_2[n-1] \rangle$	0	+	8	+	0	+	0	+	0	$= 8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-2]$	0		0		-4		8		-4	
$\langle \vec{s}_1, \vec{s}_2[n-2] \rangle$	0	+	0	+	0	+	0	+	8	$= 8$

$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-3]$	-4		0		0		-4		8	
$\langle \vec{s}_1, \vec{s}_2[n-3] \rangle$	-16	+	0	+	0	+	0	+	-16	$= -32$

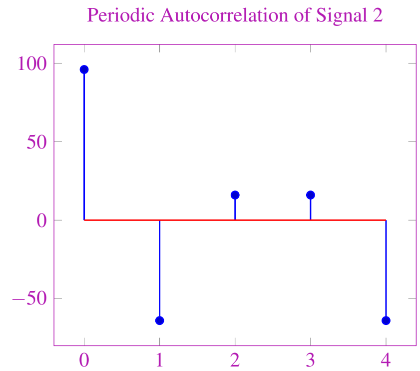
$\vec{s}_1$	4		-2		0		0		-2	
$\vec{s}_2[n-4]$	8		-4		0		0		-4	
$\langle \vec{s}_1, \vec{s}_2[n-4] \rangle$	32	+	8	+	0	+	0	+	8	$= 48$



(d) Sketch the periodic autocorrelation (correlation with itself) of signal 2 assuming a period of 5.

Answer:

The autocorrelation is as follows. Autocorrelation is a special case of cross-correlation (it is the cross-correlation of a signal with itself). See the answer for part (c) for an example of how to compute cross-correlation.


Periodic Autocorrelation of Signal 2
100
50
0
−50
0
1
2
3
4