EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 11B

1. Search and Rescue Dogs

Berkeley's Puppy Pound needs your help! While Mr. Muffin was being walked, the volunteer let go of his leash and he is now running wild in the streets of Berkeley (which are quite dangerous)! Thankfully, all of the puppies at the pound have a collar that sends a bluetooth signal to receiver towers, which are spread throughout the streets (pictured below). If the puppy/collar is within range of the receiver tower, the collar will send the tower a message: the distance of the collar to the tower. Each cell tower has a range of 3 city blocks. Can you help the pound locate their lost puppy?

Note: A city block is defined as the middle of an intersection to the middle of an adjacent intersection (scale provided on map.) Mr. Muffin is constrained to running wild in the streets, meaning he won't be found in any buildings. If your TA asks 'Where is Mr. Muffin?' it is sufficient to answer with his intersection or 'between these two intersections.'

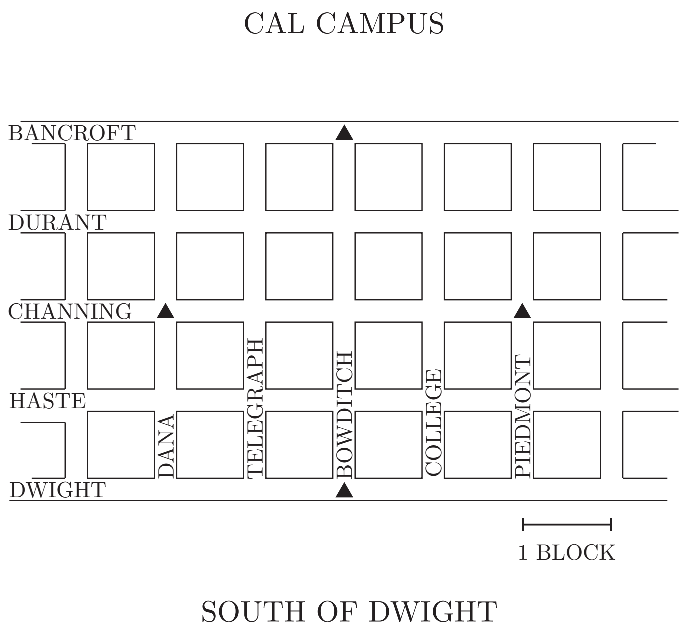

[1]

(a) You check the logs of the cell towers, and they have received the following messages:

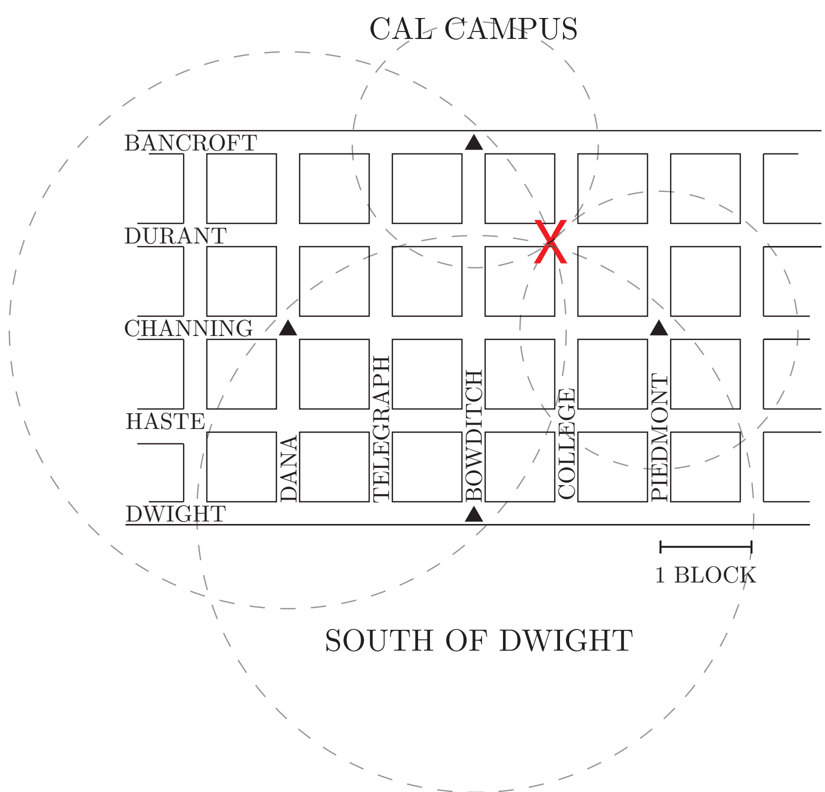
Sensor	Distance
N	1.3
W	3
E	1.5
S	3

On the map provided, identify where Mr. Muffin is!

[1] http://www.pupsmile.com/wp-content/uploads/2012/11/running_happy_dog-1024x684.jpeg



Answer:



(b) Can you set this up as a system of equations? Is it linear? If it's not linear, can you think of a way to make it linear? Now, how do you set this up in matrix form?
Hint: Set $(0,0)$ to be Channing and Bowditch.
Hint 2: Distance $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Hint 3: You don't need all 4 equations. You have two unknowns, x and y. You know from lecture that you need three circles to uniquely find a point. How can you use the third circle/equation to get two

equations and two unknowns?

Answer:

First, set up the system of equations:

$$(x-0)^2+(y-2)^2=1.3^2$$
$$(x+2)^2+(y-0)^2=3.0^2$$
$$(x-2)^2+(y-0)^2=1.5^2$$

Simplify out:

$$x^2+y^2-4y+4=1.3^2$$
$$x^2+4x+4+y^2=3.0^2$$
$$x^2-4x+4+y^2=1.5^2$$

Then subtract equation (1) from equations (2) and (3):

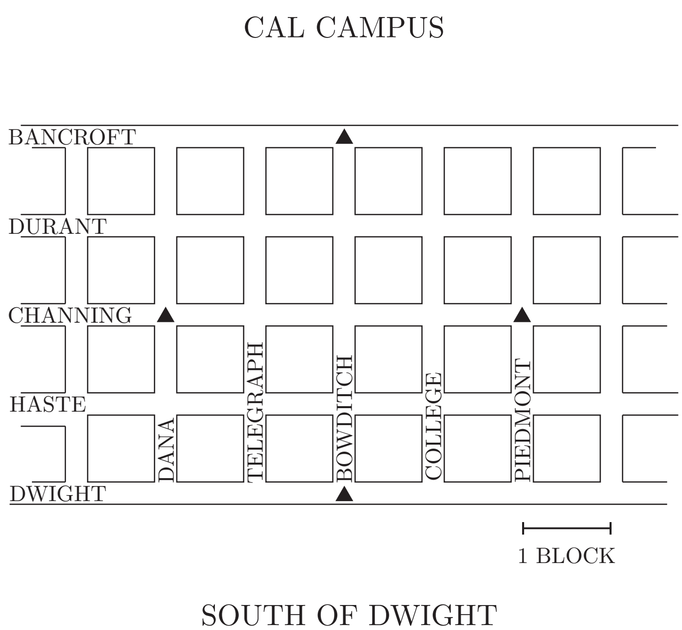
$$4x+4y=3.0^2-1.3^2$$
$$-4x+4y=1.5^2-1.3^2$$

This solves to $x=0.84, y=0.98$ which is roughly College and Durant.

(c) Suppose Mr. Muffin is moving fast, and by the time you get to destination in part (a) he's already run off! You check the logs of the cell towers again, and see the following updated messages:

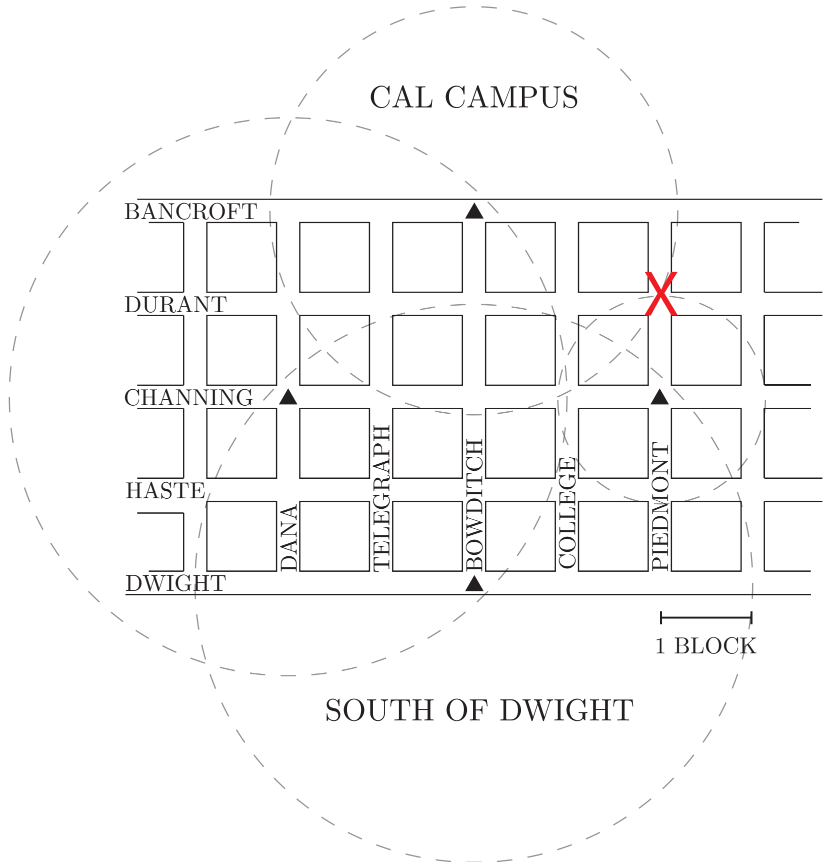
Sensor	Distance
N	2.2
W	Out of Range
E	1.1
S	Out of Range

Can you find Mr. Muffin?



Answer:

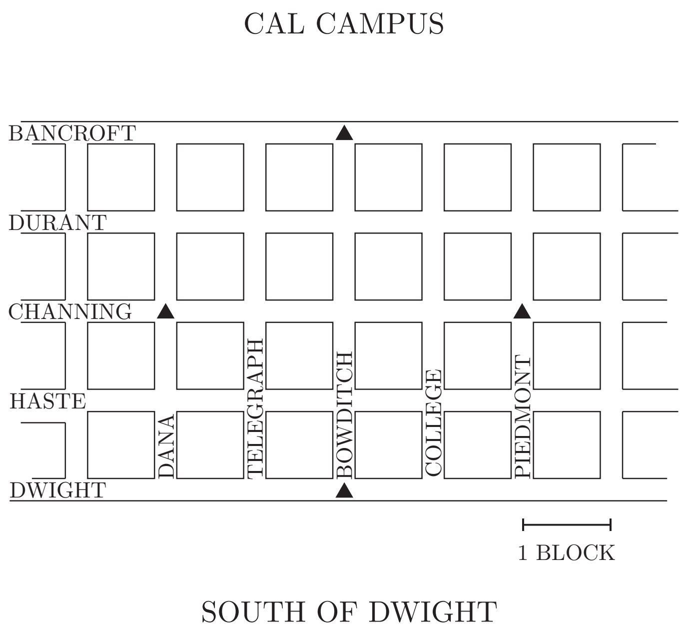
With two out of range sensors, you might think that you will not be able to find a unique solution (you need 3 circles to intersect at a point.) The trick is that out of range still provide information on where Mr. Muffin is NOT. See the diagram below.



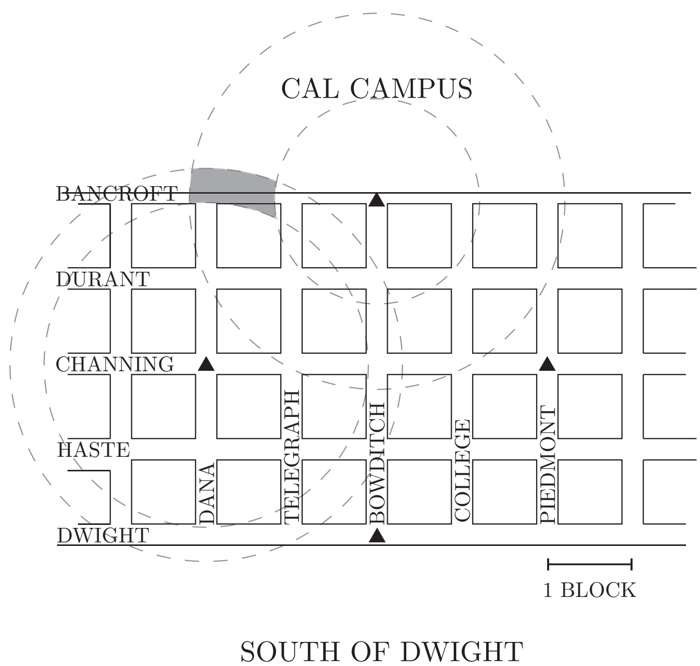
(d) Mr. Muffin is a very mischievous puppy, and while playing and running around he damaged his collar. The transmitter on his collar will still send a signal to the receiver towers, but the distance sensor has noise. You check the logs of the cell towers, and they have received the following messages:

Sensor	Distance
N	1.7 ± 0.5
W	2.1 ± 0.2
E	Out of Range
S	Out of Range

On the map provided, identify where Mr. Muffin is! Can you find exactly where he is?



Answer:
You can't find exactly where he is, but you know he is somewhere between Dana/Telegraph and Bancroft. See the diagram below.



2. A review of Inner Products

Find the inner product of the following three pairs of vectors.

(a)

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Answer: Recall that the inner product of two vectors $\vec{x}$ and $\vec{y}$ is $\vec{x}^T\vec{y}$, thus:

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = 1+3 = 4$$

(b)

$$\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$$

Answer: When working with real numbers, the inner product is commutative, thus the answer is the same as part a), 4

(c)

$$\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}-3\\2\\1\end{bmatrix}$$

Answer:

$$\begin{bmatrix}1 & 0 & 3\end{bmatrix}\begin{bmatrix}-3\\2\\1\end{bmatrix} = -3+3 = 0$$

3. From Inner Products To Projections

Given that $\langle\vec{x},\vec{y}\rangle$ is a measure of similarity between two vectors, let's try to use this to find how much of one vector $\vec{y}$ is in the direction of another vector $\vec{x}$.

(a) Let's start with $\langle\vec{x},\vec{y}\rangle$. We want a quantity that is independent of the norm of $\vec{x}$, $\|\vec{x}\|$. Is $\langle\vec{x},\vec{y}\rangle$ independent of the norm? Consider $\langle\vec{x},\vec{y}\rangle$ for the examples below.

$\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}, \vec{y} = \begin{bmatrix}2\\1\end{bmatrix}, \vec{x} = \begin{bmatrix}2\\2\end{bmatrix}, \vec{y} = \begin{bmatrix}2\\1\end{bmatrix}$

Answer:

For the first example above, we find that $\langle\vec{x},\vec{y}\rangle = 2+1 = 3$.

For the second example, we find that $\langle\vec{x},\vec{y}\rangle = 4+2 = 6$.

Notice that in both cases, the direction of the vector $\vec{x}$ did not change. However, the value of $\langle\vec{x},\vec{y}\rangle$ was dependent on the norm of $\|\vec{x}\|$, not just the direction.

(b) Suppose we divide $\langle\vec{x},\vec{y}\rangle$ by the norm of $\vec{x}$, $\|\vec{x}\|$, to get $\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|}$. Is this new quantity independent of the norm of $\vec{x}$? Test it on the examples above.

Answer:

For the first case: $\|\vec{x}\| = \sqrt{1+1} = \sqrt{2}$, so $\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|} = \frac{3}{\sqrt{2}}$.

For the second case: $\|\vec{x}\| = \sqrt{4+4} = 2\sqrt{2}$, so $\frac{\langle\vec{x},\vec{y}\rangle}{\|\vec{x}\|} = \frac{6}{2\sqrt{2}} = \frac{3}{\sqrt{2}}$.

Notice here that we have removed the dependence on the norm of $\vec{x}$.

(c) We now have a scalar quantity that represents how much of $\vec{y}$ is in the direction of $\vec{x}$. Let's try to find a vector that is how much of $\vec{y}$ is in the $\vec{x}$ direction. That is, we are looking for a vector $\vec{z}$ that has a norm of $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|}$ and points in the same direction as $\vec{x}$.

Answer:

If we multiply our answer from the last part by the vector $\vec{x}$, we get a quanity that point in the correct direction. However, its norm is $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|}\|\vec{x}\| = \langle \vec{x}, \vec{y} \rangle$. In order to get the expected norm, we need to scale our vector $\vec{x}$ by $\frac{1}{\|\vec{x}\|}$.

Thus, we find $\vec{z} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|^2}\vec{x}$. This is also known as the projection. We will use it a lot in the upcoming dicussions on least squares.

(d) Given the projection between two vectors, defined as $\text{proj}_{\vec{x}}\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|^2}\vec{x}$, prove the Cauchy-Schwarz inequality, $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

Answer:

Let the vector $\vec{p}$ be the projection of $\vec{y}$ on the vector $\vec{x}$, that is, $\vec{p} = \text{proj}_{\vec{x}}\vec{y} = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|^2}\vec{x}$.

We know that the magnitude of the projection of $\vec{y}$ onto $\vec{x}$ must be less than or equal to the magnitude of $\vec{y}$.

$$\|\vec{y}\| \geq \|\vec{p}\|$$
$$\|\vec{y}\| \geq \left|\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|^2}\right|\|\vec{x}\|$$
$$\|\vec{y}\| \geq \frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{x}\|}$$
$$\|\vec{y}\|\|\vec{x}\| \geq |\langle \vec{x}, \vec{y} \rangle|$$

(e) Consider the quantity $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|}$. What is the maximum this quantity could be? When does this occur? What is the minimum this quantity could be? When does this occur?

Answer:

Using the Cauchy-Schwarz inequality, we know that $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\|\|\vec{y}\|$.

$$\frac{|\langle \vec{x}, \vec{y} \rangle|}{\|\vec{x}\|\|\vec{y}\|} \leq 1 \implies -1 \leq \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|} \leq 1$$

The maximum occurs when $\vec{x}$ and $\vec{y}$ are parallel. The quantity becomes $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|} = 1$.

The minimum occurs when $\vec{x}$ and $\vec{y}$ are antiparallel. The quantity becomes $\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|} = -1$.

(f) We define the angle between two vectors as $\cos(\theta) = \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\|\|\vec{y}\|}$. When do two vectors have an angle of $90°$ between them? When do they have an angle of $0°$? When do they have an angle of $180°$?

Answer:

Vectors have an angle of $90°$ when their inner product is zero. The have an angle of $0°$ when they point in the same direction and an angle of $180°$ when they point in opposite directions.

