EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 14A

1. Mechanical Gram-Schmidt (Fall 2016 Final)

(a) Use Gram-Schmidt to find an orthonormal basis for the following three vectors.

$$\vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

Answer: We summarize the steps of the Gram-Schmidt algorithm as follows:

i. $\vec{u}_1' = \vec{v}_1; \qquad \vec{u}_1 = \frac{\vec{u}_1'}{\|\vec{u}_1'\|}$.

ii. $\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1; \qquad \vec{u}_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|}$.

iii. $\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2; \qquad \vec{u}_3 = \frac{\vec{u}_3'}{\|\vec{u}_3'\|}$.

For the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$, this is

i. $\vec{u}_1' = \vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}$.

$\vec{u}_1 = \frac{\vec{u}_1'}{\|\vec{u}_1'\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$.

ii. $\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{u}_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

iii. $\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} + \sqrt{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

$\vec{u}_3 = \frac{\vec{u}_3'}{\|\vec{u}_3'\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$

A valid basis $\mathscr{B}$ is given by:

$$\mathscr{B} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}$$

(b) Express $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ as vectors in the basis you found in part (a).

Answer:
$\vec{v}_1 = \vec{u}_1' = \|\vec{u}_1'\|\vec{u}_1 = 2\vec{u}_1$
$\vec{v}_2 = \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 + \vec{u}_2' = \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 + \|\vec{u}_2'\|\vec{u}_2 = 0\vec{u}_1 + \vec{u}_2$
$\vec{v}_3 = \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 + \vec{u}_3' = \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 + \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 + \|\vec{u}_3'\|\vec{u}_3 = \vec{u}_1 - \sqrt{2}\vec{u}_2 + \vec{u}_3$
Using the basis above:

$$[\vec{v}_1]_{\mathscr{B}} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, [\vec{v}_2]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\vec{v}_3]_{\mathscr{B}} = \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

2. Cactus Care (30 points) (Spring 2018 final)

You want to monitor the light received by your cactus over the course of the day, when you aren't around. You design a transmitter that sends the following periodic code of length $N = 5$:

$$\vec{c} = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

You can encode information about the light by multiplying the code with the light intensity (l). With your cell phone, you received $\vec{r}$, which is a shifted version of the code $\vec{c}l$.

(a) (4 points) Write a matrix A such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except one entry which contains the light intensity l. For example, if $\vec{r} = \begin{bmatrix} 2l & l & -3l & 2l & l \end{bmatrix}^T$ i.e. $\vec{c}l$ is shifted by 1, $\vec{y} = \begin{bmatrix} 0 & l & 0 & 0 & 0 \end{bmatrix}^T$. If shifted by 3, $\vec{y} = \begin{bmatrix} 0 & 0 & 0 & l & 0 \end{bmatrix}^T$.

Answer: A is a circulant matrix containing all of the possible shifts of $\vec{c}$ in its columns:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & -3 \\ -3 & 1 & 2 & 1 & 2 \\ 2 & -3 & 1 & 2 & 1 \\ 1 & 2 & -3 & 1 & 2 \\ 2 & 1 & 2 & -3 & 1 \end{bmatrix}$$

You could also write this with the shift notation we used in class, where $\vec{c}^{(k)}$ is $\vec{c}$ circularly shifted by k.

$$A = \begin{bmatrix} \vec{c}^{(0)} & \vec{c}^{(1)} & \vec{c}^{(2)} & \vec{c}^{(3)} & \vec{c}^{(4)} \end{bmatrix}$$

(b) (6 points) This semester your learned several techniques for solving linear systems of equations. For each of the following techniques, could you use it to solve the matrix equation from Part A? Justify your answer in 1-2 sentences. Assume there is no noise.

Gaussian Elimination ◯ yes ◯ no
Explain:
Answer: Yes.
A is a square matrix with linearly independent rows, so we can use Gaussian elimination to solve for $\vec{y}$.

Least Squares ◯ yes ◯ no
Explain:
Answer: Yes.

Least squares can be used to solve when there are at least as many rows as columns. In this case, where A is square and invertible, the least squares solution is the same as the one you would get with Gaussian elimination.

Orthogonal Matching Pursuit ◯ yes ◯ no
Explain:
Answer: Yes.
Orthogonal Matching Pursuit can be used for solving for vectors that are mostly zero (sparse vectors). Since $\vec{y}$ contains all zeros except one element, it is sparse and we can use OMP. Alternative justification: OMP can be used to extract the messages from a small number of beacons sending periodic signals, which is what is happening in this problem.

You set up another light detector in the lab to see if it's better. The two light detectors has a transmitter with a different **periodic** codes: $\vec{c}_1$ and $\vec{c}_2$.

$$\vec{c}_1 = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

$$\vec{c}_2 = \begin{bmatrix} 3 & 1 & 2 & -2 & -1 \end{bmatrix}^T$$

As before, the codes are multiplied by the light intensities at each location, l_1 and l_2, and your cell phone receives the sum of shifted codes, each weighted by the light at that location.

(c) (5 points) Write a new matrix A such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except two entries which contain l_1 and l_2.
Hint: The positions of l_1 and l_2 in the vector $\vec{y}$ will depend on the unknown shifts in $\vec{c}_1$ and $\vec{c}_2$, respectively. For example, $\vec{y} = \begin{bmatrix} 0 & l_1 & 0 & 0 & 0 & 0 & 0 & l_2 & 0 & 0 \end{bmatrix}^T$, if $\vec{c}_1$ and $\vec{c}_2$ are shifted by 1 and 2 respectively.
Answer: A contains all of the possible shifts of each code in its columns.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & -3 & 3 & -1 & -2 & 2 & 1 \\ -3 & 1 & 2 & 1 & 2 & 1 & 3 & -1 & -2 & 2 \\ 2 & -3 & 1 & 2 & 1 & 2 & 1 & 3 & -1 & -2 \\ 1 & 2 & -3 & 1 & 2 & -2 & 2 & 1 & 3 & -1 \\ 2 & 1 & 2 & -3 & 1 & -1 & -2 & 2 & 1 & 3 \end{bmatrix}$$

You could also write this with the shift notation we used in class, where $\vec{c}^{(k)}$ is $\vec{c}$ circularly shifted by k.

$$A = \begin{bmatrix} \vec{c}_1^{(0)} & \vec{c}_1^{(1)} & \vec{c}_1^{(2)} & \vec{c}_1^{(3)} & \vec{c}_1^{(4)} & \vec{c}_2^{(0)} & \vec{c}_2^{(1)} & \vec{c}_2^{(2)} & \vec{c}_2^{(3)} & \vec{c}_2^{(4)} \end{bmatrix}$$

(d) (6 points) For each of the following techniques, could you use it to solve the matrix equation from Part D, with two different light sensors? Justify your answer in 1-2 sentences. Assume there is no noise.
Gaussian Elimination ◯ yes ◯ no
Explain:
Answer: No.
There are more columns than rows in A so if we attempt Gaussian Elimination, there will not be a pivot in every column. This means there are infinitely many possible solutions.

Least Squares ◯ yes ◯ no
Explain:
Answer: No.
Least squares can only be used to solve a system of equations when there are at least as many rows as columns. In this case there are fewer rows than columns, so we cannot use least squares.
To see this, let's look at the least squares equation:

$$\hat{\vec{y}} = (A^T A)^{-1} A^T \vec{r}$$

If A has more columns than rows, $A^T A$ cannot be full rank, so it is not invertible.

Orthogonal Matching Pursuit ◯ yes ◯ no
Explain:
Answer: Yes.
Orthogonal Matching Pursuit can be used for solving for vectors that are mostly zeros, even when the system of equations is underdetermined. Since $\vec{y}$ contains all zeros except two elements, it is sparse and we can use OMP. Alternative justification: OMP can be used to extract the messages from a small number of beacons sending periodic signals, which is what is happening in this problem.

(e) (3 points) In order to judge if your codes are "good", you want to calculate the autocorrelations and cross-correlation of your codes. Professor Waller helps you calculate the following:

$$\text{autocorrelation of } \vec{c}_1 = \begin{bmatrix} 19 & -3 & \textbf{??} & -2 & -3 \end{bmatrix}^T$$
$$\text{autocorrelation of } \vec{c}_2 = \begin{bmatrix} 19 & 0 & -5 & -5 & 0 \end{bmatrix}^T$$
$$\text{cross-correlaton of } \vec{c}_1 \text{ with } \vec{c}_2 = \begin{bmatrix} 0 & -10 & 12 & 11 & -4 \end{bmatrix}^T$$

Finish the set by calculating the unknown term in the autocorrelation of c_1.
Answer:
The missing term of the autocorrelation is the inner product of the code with a version of itself, circularly shifted by 2.

$$\vec{c}_1 = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

$$\text{autocorr. at lag 2 } = (1)(2) + (-3)(1) + (2)(2) + (1)(1) + (2)(-3) = -2$$

(f) (6 points) Consider the following set of codes (c_3 and c_4).

$$\vec{c}_3 = \begin{bmatrix} 1 & -2 & -3 & 2 & 1 \end{bmatrix}^T \qquad \vec{c}_4 = \begin{bmatrix} 1 & 1 & 2 & -2 & -3 \end{bmatrix}^T$$
$$\text{autocorr. of } \vec{c}_3 = \begin{bmatrix} 19 & 1 & -10 & -10 & 1 \end{bmatrix}^T$$
$$\text{autocorr. of } \vec{c}_4 = \begin{bmatrix} 19 & 2 & -11 & -11 & 2 \end{bmatrix}^T$$
$$\text{cross-correlation of } \vec{c}_3 \text{ with } \vec{c}_4 = \begin{bmatrix} -14 & -16 & 5 & 18 & -2 \end{bmatrix}^T$$

If you use OMP to solve for the light intensities, which set of codes (c_1,c_2 OR c_3,c_4) is more robust to noise in the received signal? Justify your answer. For the set of codes that is worse, what mistake will is most likely to happen during the OMP algorithm in the presence of noise?

◯ c_1, c_2 are more robust ◯ c_3, c_4 are more robust

Answer: $\mathbf{c_1}$ **and** $\mathbf{c_2}$ are more robust to noise. This is because they are "more orthogonal" than c_3 and c_4 for all possible shifts, ie. the autocorrelation (at non-zero shift) and cross-correlations of c_1, c_2 are generally closer to zero.

Specifically, the cross-correlation of c_3, c_4 has a peak of magnitude 18, which is almost as high as the autocorrelation at zero shift. This means that if we try to use OMP with c_3, c_4 we are likely to accidentally mistake c_3 for c_4 at a different shift (or vice versa).