
EECS 16A Designing Information Devices and Systems I
 Spring 2019 Discussion 2A

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 9 & 5 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 5 & 5 & 8 \\ 6 & 1 & 2 \\ 4 & 1 & 7 \\ 3 & 2 & 2 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \quad \mathbf{F} = \begin{bmatrix} 5 & 3 & 4 \\ 1 & 8 & 2 \\ 2 & 3 & 5 \end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a) **AB**

Answer: Since both **A** and **B** are 2×2 matrices, the product exists and is a 2×2 matrix.

$$\mathbf{AB} = \begin{bmatrix} 11 & 6 \\ 12 & 7 \end{bmatrix}.$$

(b) **BA**

Answer: Since both **A** and **B** are 2×2 matrices, the product exists and is a 2×2 matrix.

$$\mathbf{BA} = \begin{bmatrix} 7 & 18 \\ 4 & 11 \end{bmatrix}.$$

(c) **CD**

Answer: Since **C** is a 2×4 matrix and **D** is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{CD} = \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}.$$

(d) **DC**

Answer: Since **C** is a 2×4 matrix and **D** is a 4×3 matrix, the product does not exist. This is because the number of columns in the first matrix (**D**) should match the number of rows in the second matrix (**C**) for this product to be defined.

(e) **EF**

Answer: Since **E** and **F** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(f) **FE**

Answer: Since **E** and **F** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{FE} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a “rotation matrix,” we will see it “rotate” in the true sense here. Similarly, when we multiply a vector by a “reflection matrix,” we will see it be “reflected.” The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!

Part 1: Rotation Matrices as Rotations

- (a) We are given matrices \mathbf{T}_1 and \mathbf{T}_2 , and we are told that they will rotate the unit square by 15° and 30° , respectively. Design a procedure to rotate the unit square by 45° using only \mathbf{T}_1 and \mathbf{T}_2 , and plot the result in the IPython notebook. How would you rotate the square by 60° ?

Answer:

Apply \mathbf{T}_1 and \mathbf{T}_2 in succession to rotate the unit square by 45° . To rotate the square by 60° , you can either apply \mathbf{T}_2 twice, or if you prefer variety, apply \mathbf{T}_1 twice and \mathbf{T}_2 once.

- (b) Try to rotate the unit square by 60° using only one matrix. What does this matrix look like?

Answer: This matrix will look like the rotation matrix that rotates a vector by 60° . This matrix can be composed by multiplying \mathbf{T}_1 by \mathbf{T}_1 by \mathbf{T}_2 (or equivalently, \mathbf{T}_2 by \mathbf{T}_2).

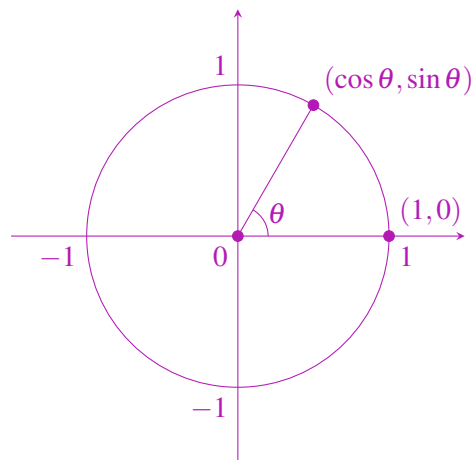
- (c) \mathbf{T}_1 , \mathbf{T}_2 , and the matrix you used in part (b) are called “rotation matrices.” They rotate any vector by an angle θ . Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

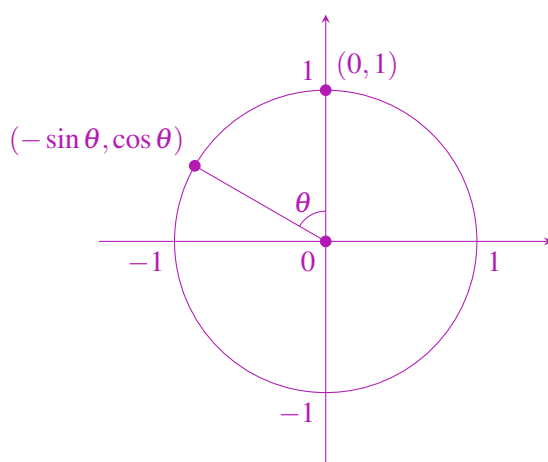
where θ is the angle of rotation. (*Hint: Use your trigonometric identities!*)

Answer:

Let’s try to derive this matrix using trigonometry. Suppose we want to rotate the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by θ .



We can use basic trigonometric relationships to see that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated by θ becomes $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$. Similarly, rotating the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ by θ becomes $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$:



We can also scale these pre-rotated vectors to any length we want, $\begin{bmatrix} x \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ y \end{bmatrix}$, and we can observe graphically that they rotate to $\begin{bmatrix} x \cos \theta \\ x \sin \theta \end{bmatrix}$ and $\begin{bmatrix} -y \sin \theta \\ y \cos \theta \end{bmatrix}$, respectively. Rotating a vector solely in the x -direction produces a vector with both x and y components, and, likewise, rotating a vector solely in the y -direction produces a vector with both x and y components.

Finally, if we want to rotate an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$, we can combine what we derived above. Let x' and y' be the x and y components after rotation. x' has contributions from both x and y : $x' = x \cos \theta - y \sin \theta$. Similarly, y' has contributions from both components as well: $y' = x \sin \theta + y \cos \theta$. Expressing this in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we've derived the 2-dimensional rotation matrix.

Alternative solution:

The reason the matrix is called a rotation matrix is because it transforms the unit vector $\begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix}$ to give $\begin{bmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{bmatrix}$.

Proof:

$$\begin{aligned} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} &= \cos \alpha \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + \sin \alpha \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos \alpha \cos \theta - \sin \alpha \sin \theta \\ \cos \alpha \sin \theta + \sin \alpha \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha + \theta) \\ \sin(\alpha + \theta) \end{bmatrix} \end{aligned}$$

- (d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? *Don't use inverses!* (**Note:** We do not expect you to know inverses at this point; we will cover them soon.)

Answer:

Use a rotation matrix that rotates by -60° .

$$\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$

- (e) Use part (d) to obtain the “inverse” rotation matrix for a matrix that rotates a vector by θ . Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

Answer:

The inverse matrix is as follows:

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

We can see from this inverse matrix that the product of the rotation matrix and its inverse is the identity matrix.

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- (f) What are the matrices that reflect a vector about the (i) x -axis, (ii) y -axis, and (iii) $x = y$

Answer:

The matrix that reflects about the x -axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix that reflects about the y -axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrix that reflects about $x = y$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

[Practice Problem] Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Follow your TA to obtain the answers to the following questions!

- (a) Let’s see what happens to the unit square when we rotate the square by 60° and then reflect it along the y -axis.
- (b) Now, let’s see what happens to the unit square when we first reflect the square along the y -axis and then rotate it by 60° .

Answer: (For parts (a) and (b)): The two operations are not the same.

- (c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

Answer:

The resulting matrices that are obtained (by multiplying the two matrices) are different depending on the order of multiplication.

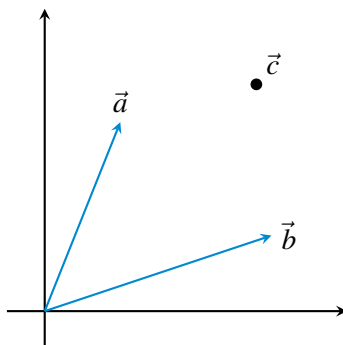
- (d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?

Answer:

It turns out that reflections are not commutative unless the two reflection axes are perpendicular to each other. For example, if you reflect about the x -axis and the y -axis, it is commutative. But if you reflect about the x -axis and $x = y$, it is not commutative.

3. Visualizing Span

We are given a point \vec{c} that we want to get to, but we can only move in two directions: \vec{a} and \vec{b} . We know that to get to \vec{c} , we can travel along \vec{a} for some amount α , then change direction, and travel along \vec{b} for some amount β . We want to find these two scalars α and β , such that we reach point \vec{c} . That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.



- (a) First, consider the case where $\vec{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{c} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Draw these vectors on a sheet of paper. Now find the two scalars α and β , such that we reach point \vec{c} . What are these scalars if we use $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead?

Answer: First set: $\alpha = -4, \beta = 2$

Second set: $\alpha = 6, \beta = -4$

- (b) Now formulate the general problem as a system of linear equations and write it in matrix form.

Answer:

$$\begin{cases} \alpha a_1 + \beta b_1 = c_1 \\ \alpha a_2 + \beta b_2 = c_2 \end{cases}$$

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$