EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 3A

1. Mechanical Inverses

In each part, determine whether the inverse of $\mathbf{A}$ exists. If it exists, find it.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{9}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$.

(b) $\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{cc|cc} 5 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 5 & 4 & 1 & 0 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow -5R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -5 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & -1 & 1 & -5 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow -R_2} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -4 \\ 0 & 1 & -1 & 5 \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$.

(c) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 0 & 4 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{1}{5}R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \quad \overset{R_2 \leftrightarrow R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & -1 & 1 & -\frac{1}{5} & 0 & 1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow -R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \end{array}\right] \quad \overset{R_3 \leftarrow -R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{5} & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_2 \leftarrow R_2 + R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right] \quad \overset{R_1 \leftarrow R_1 - 3R_3}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{5} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

$$\overset{R_1 \leftarrow R_1 - R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{4}{5} & 2 & 1 \\ 0 & 1 & 0 & \frac{2}{5} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{2} & 0 \end{array}\right]$$

Therefore, we get $\mathbf{A}^{-1} = \begin{bmatrix} -\frac{4}{5} & 2 & 1 \\ \frac{2}{5} & -\frac{1}{2} & -1 \\ \frac{1}{5} & -\frac{1}{2} & 0 \end{bmatrix}$.

(d) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

Answer:

We apply the Gauss-Jordan method:

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \quad \overset{R_1 \leftarrow \frac{1}{5}R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_2 \leftarrow \frac{1}{2}R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 1 & 1 & 2 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \quad \overset{R_2 \leftarrow R_2 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{R_3 \leftarrow R_3 - R_1}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right] \quad \overset{R_3 \leftarrow R_3 + R_2}{\Longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{5} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right]$$

While row-reducing, we notice that the second column doesn't have a pivot (and that there is also a row of zeros). Therefore, no inverse exists.

Reference Definitions

Vector spaces: A *vector space* V is a set of elements that is 'closed' under vector addition and scalar multiplication and contains a zero vector. What does closed mean?

That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

More formally, a *vector space* (V, F) is a set of vectors V, a set of scalars F, and two operators that satisfy the following properties:

- Vector Addition

- Associative: $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w} \in V$.
- Commutative: $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ for any $\vec{v},\vec{u} \in V$.
- Additive Identity: There exists an additive identity $\vec{0} \in V$ such that $\vec{v}+\vec{0} = \vec{v}$ for any $\vec{v} \in V$.
- Additive Inverse: For any $\vec{v} \in V$, there exists $-\vec{v} \in V$ such that $\vec{v}+(-\vec{v}) = \vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.

- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ for any $\vec{v} \in V$, $\alpha,\beta \in F$.
  - Multiplicative Identity: There exists $1 \in F$ where $1 \cdot \vec{v} = \vec{v}$ for any $\vec{v} \in F$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u}+\alpha\vec{v}$ for any $\alpha \in F$ and $\vec{u},\vec{v} \in V$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta \in F$ and $\vec{v} \in V$.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the above conditions (closure under vector addition and scalar multiplication and existence of a zero vector) hold for the elements in the subspace W.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

2. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? What about for some subspace of $\mathbb{R}^3$?

$$V_1 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

Answer:

- V_1: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$.
- V_2: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$.
- V_3: No, $\vec{v}_2+\vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent.

3. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c,d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

Answer:

Yes, V is a subspace of $\mathbb{R}^3$. We will *prove this* by using the definition of a subspace.

First of all, note that V is a subset of $\mathbb{R}^3$ – all elements in V are of the form $\begin{bmatrix}c+d\\c\\c+d\end{bmatrix}$, which is a 3-dimensional real vector.

Now, consider two elements $\vec{v}_1,\vec{v}_2 \in V$ and $\alpha \in \mathbb{R}$.

This means that there exists $c_1, d_1 \in \mathbb{R}$, such that $\vec{v}_1 = c_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_1 \begin{bmatrix}1\\0\\1\end{bmatrix}$. Similarly, there exists $c_2, d_2 \in \mathbb{R}$, such that $\vec{v}_2 = c_2 \begin{bmatrix}1\\1\\1\end{bmatrix} + d_2 \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Now, we can see that

$$\vec{v}_1 + \vec{v}_2 = (c_1 + c_2) \begin{bmatrix}1\\1\\1\end{bmatrix} + (d_1 + d_2) \begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\vec{v}_1 + \vec{v}_2 \in V$.

Also,

$$\alpha \vec{v}_1 = (\alpha c_1) \begin{bmatrix}1\\1\\1\end{bmatrix} + (\alpha d_1) \begin{bmatrix}1\\0\\1\end{bmatrix},$$

so $\alpha \vec{v}_1 \in V$.

Furthermore, we observe that the zero vector is contained in V, when we set $c = 0$ and $d = 0$.

We have thus shown both of the no escape (closure) properties and the existence of a zero vector, so V is a subspace of $\mathbb{R}^3$.

