EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 3B

1. Constructing a Basis

Let's consider a subspace of $\mathbb{R}^3$ called V which has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V?

Answer:

Any vector $\vec{v}$ in V is going to look as follows:

$$\vec{v} = \begin{bmatrix} 2(a_2 + a_3) \\ a_2 \\ a_3 \end{bmatrix} = a_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Now, we consider the set of vectors $\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. The vectors are linearly independent. Furthermore, from the above equation, any vector $\vec{v} \in V$ can be expressed as a linear combination of the vectors in $\mathscr{B}$ (the corresponding coefficients are a_2 and a_3). This means that $V = \text{span}\{\mathscr{B}\}$.

Therefore,

$$\mathscr{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for V.

$\dim(\mathscr{B}) = 2$ (there are two vectors in $\mathscr{B}$), so the dimension of V is 2.

2. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$ and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. The dimension of $\mathbb{R}^m$ is m, so you would need m linearly independent vectors to describe the vector space. Since $n < m$, this is not possible.

(b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

Answer:

Yes, this is possible. The only condition we need is that $\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ is linearly independent. If the vectors are linearly independent, since there are m of them, they will span $\mathbb{R}^m$.

(c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

Answer:

No, $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ cannot form a basis for $\mathbb{R}^m$. $\mathbb{R}^m$ will be spanned by m linearly independent vectors. Any additional vectors in $\mathbb{R}^m$ must already exist in the span of the previous vectors, and are therefore linearly dependent. Since $n > m$, some of the vectors have to be linearly dependent, so they cannot form a basis.

The two regimes—one where $n > m$ and one where $n < m$—give rise to two different classes of interesting problems. You might learn more about them in upper division courses!

3. Exploring Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis for $\mathbb{R}^2$ (or $\mathbb{R}^3$ for (e)? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

The matrix is already row reduced. The column spaces of the row reduced matrix and the original matrix are the same.

Not a basis for $\mathbb{R}^2$.

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

Answer:

Column space: $\mathbb{R}^2$

Null space: $\text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

The two column spaces are the same as the column span $\mathbb{R}^2$.

This is a basis for $\mathbb{R}^2$.

(d) $\begin{bmatrix}-2 & 4\\3 & -6\end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix}1\\-\frac{3}{2}\end{bmatrix}\right\}$

Null space: $\text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^2$.

(e) $\begin{bmatrix}1 & 2 & 1\\-1 & 0 & 3\\0 & -1 & -2\end{bmatrix}$

Answer:

Column space: $\text{span}\left\{\begin{bmatrix}1\\0\\-\frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\1\\-\frac{1}{2}\end{bmatrix}\right\}$.

Null space: $\text{span}\left\{\begin{bmatrix}3\\-2\\1\end{bmatrix}\right\}$

The two column spaces are not the same.

Not a basis for $\mathbb{R}^3$.