EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 4B

Reference Definitions: Matrices and Linear (In)Dependence We've seen that the following statements are equivalent for an $n \times n$ matrix $\mathbf{A}$, meaning, if one is true then all are true:

- $\mathbf{A}$ is invertible
- The equation $\mathbf{A}\vec{x} = \vec{0}$, has a unique solution, which is $\vec{x} = \vec{0}$
- The columns of $\mathbf{A}$ are linearly independent
- For each column vector $\vec{b} \in \mathbb{R}^n$, $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution $\vec{x}$
- $\text{Null}(\mathbf{A}) = \vec{0}$

Conversely, if one of the following is true, then all of the following are true:

- $\mathbf{A}$ is not invertible
- $\mathbf{A}\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$
- The columns of $\mathbf{A}$ are linearly dependent
- There is not be a unique $\vec{x}$ for every $\vec{b}$ where $\mathbf{A}\vec{x} = \vec{b}$
- $\text{Null}(\mathbf{A})$ contains more than just the zero vector $\vec{0}$

These are part of what is known as the Invertible Matrix Theorem.

1. **Mechanical Eigenvalues and Eigenvectors**

In each part, find the eigenvalues of the matrix $\mathbf{M}$ and the associated eigenvectors.

(a) $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 9-\lambda \end{bmatrix}\right) = 0$$

The determinant of a diagonal matrix is the product of the entries.

$$(1-\lambda)(9-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 1$ and $\lambda = 9$.

For the eigenvalue $\lambda = 1$:

$$(\mathbf{M} - 1\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - 1\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix}0 & 0\\0 & 8\end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_2 = 0$ or equivalently $\begin{bmatrix}x_1\\0\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 9$:

$$(\mathbf{M} - 9\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - 9\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}1 & 0\\0 & 9\end{bmatrix} - \begin{bmatrix}9 & 0\\0 & 9\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix}-8 & 0\\0 & 0\end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$ or equivalently $\begin{bmatrix}0\\x_2\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$.

(b) $\mathbf{M} = \begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) = \det\left(\begin{bmatrix}1-\lambda & 1\\2 & 2-\lambda\end{bmatrix}\right) = 0$$
$$(1-\lambda)(2-\lambda) - 2 = \lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$ and $\lambda = 3$.

For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$
$$\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix}\vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = -x_1$ or equivalently $\begin{bmatrix}x_1\\-x_1\end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} - 3\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix}1 & 1\\2 & 2\end{bmatrix} - \begin{bmatrix}3 & 0\\0 & 3\end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix}-2 & 1\\2 & -1\end{bmatrix}\vec{x} = \vec{0}$$

which is simply x_1 is free and $x_2 = 2x_1$ or equivalently $\begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

(c) **(PRACTICE)** $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0$$

Without changing the determinant, we can subtract $\frac{3}{\lambda}$ of row 1 from row 2.

$$\det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ -3 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 9 \\ 0 & 0 & 3-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda(4-\lambda)(3-\lambda) = 0$$

From the above equation, we know that the eigenvalues are $\lambda = 0$, $\lambda = 3$, and $\lambda = 4$.

For the eigenvalue $\lambda = 0$:

$$(\mathbf{M} - 0\mathbf{I})\vec{x} = \mathbf{M}\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

which is simply $x_3 = 0$, x_2 is free, and $x_1 = \frac{4}{3}x_2$ or equivalently $\begin{bmatrix} \frac{4}{3}x_2 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = 3$:

$$(\mathbf{M} - 3\mathbf{I})\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -3 & 0 & 0 \\ -3 & 1 & 9 \\ 0 & 0 & 0 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = 0$, x_3 is free, and $x_2 = -9x_3$ or equivalently $\begin{bmatrix} 0 \\ -9x_3 \\ x_3 \end{bmatrix}$ or equivalently span $\left\{ \begin{bmatrix} 0 \\ -9 \\ 1 \end{bmatrix} \right\}$.

For the eigenvalue $\lambda = 4$:

$$(\mathbf{M} - 4\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & 0 & 0 \\ -3 & 4 & 9 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -4 & 0 & 0 \\ -3 & 0 & 9 \\ 0 & 0 & -1 \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = x_3 = 0$ and $x_2 = x_2$ or equivalently $\begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$.

(d) **(PRACTICE) M =** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Answer:

Let's begin by finding the eigenvalues:

$$\det\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = 0$$

Without changing the determinant, we can add $\frac{1}{\lambda}$ of row 1 to row 2.

$$\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & -1 \\ 0 & -\lambda - \frac{1}{\lambda} \end{bmatrix}\right) = 0$$
$$-\lambda\left(-\lambda - \frac{1}{\lambda}\right) = \lambda^2 + 1 = 0$$

From the above equation, we know that the eigenvalues are $\lambda = i$ and $\lambda = -i$.

For the eigenvalue $\lambda = i$:

$$(\mathbf{M} - i\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = ix_2$ and x_2 is free or equivalently $\begin{bmatrix} ix_2 \\ x_2 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$.

For the eigenvalue $\lambda = -i$:

$$(\mathbf{M} + i\mathbf{I})\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\right)\vec{x} = \vec{0}$$
$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\vec{x} = \vec{0}$$

which is simply $x_1 = -ix_2$ and x_2 is free or equivalently $\begin{bmatrix} -ix_2 \\ x_2 \end{bmatrix}$ or equivalently span $\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\}$.

2. Steady State Reservoir Levels

We have 3 reservoirs: A, B and C. The pumps system between the reservoirs is depicted in Figure 1.

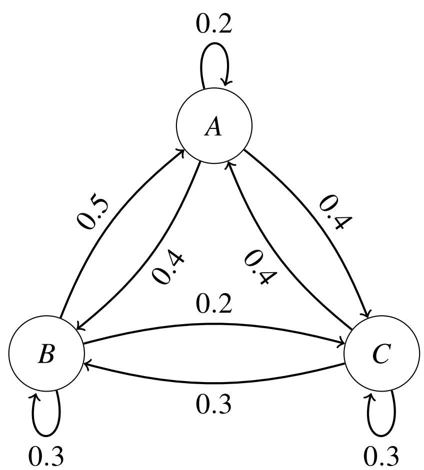

Figure 1: Reservoir pumps system.

(a) Write out the transition matrix representing the pumps system.

Answer:

$$\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix}$$

(b) Assuming that you start the pumps with the water levels of the reservoirs at $A_0 = 129, B_0 = 109, C_0 = 0$ (in kiloliters), what would be the steady state water levels (in kiloliters) according to the pumps system described above?

Hint: If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs (in kiloliters), what happens if you fill the reservoirs A, B and C with A_{ss}, B_{ss} and C_{ss} kiloliters of water, respectively, and apply the pumps once?

Hint II: Note that the pumps system preserves the total amount of water in the reservoirs. That is, no water is lost or gained by applying the pumps.

Answer:

If $\vec{x}_{ss} = \begin{bmatrix} A_{ss} \\ B_{ss} \\ C_{ss} \end{bmatrix}$ is a vector describing the steady state levels of water in the reservoirs, then we know that $\mathbf{T}\vec{x}_{ss} = 1 \cdot \vec{x}_{ss}$—that is, applying the pumps one more time wouldn't change the level of water in any of the reservoirs. This means that $\vec{x}_{ss}$ is an eigenvector of $\mathbf{T}$ associated with the eigenvalue $\lambda = 1$. Therefore,

$$\vec{x}_{ss} \in \text{Null}(\mathbf{T} - 1 \cdot \mathbf{I}) = \text{Null}\left(\begin{bmatrix} 0.2 & 0.5 & 0.4 \\ 0.4 & 0.3 & 0.3 \\ 0.4 & 0.2 & 0.3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}\right)$$

We calculate the null space of the matrix $\begin{bmatrix} -0.8 & 0.5 & 0.4 \\ 0.4 & -0.7 & 0.3 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}$, which is simply span $\left\{\begin{bmatrix} 43 \\ 40 \\ 36 \end{bmatrix}\right\}$, which means that our steady state reservoirs levels vector is of the form $\begin{bmatrix} 43\alpha \\ 40\alpha \\ 36\alpha \end{bmatrix}$, $\alpha \in \mathbb{R}$.

Furthermore, we know that the pumps system conserves the water, i.e., no water is lost by running the pumps system. Therefore, we know that the total amount of water in the reservoirs at any point in time will be $129 + 109 + 0 = 238$ (equal to the original total amount of water in the system). Therefore, we are looking for an eigenvector whose components sum to 238. In other words, we are looking for α such that $43\alpha + 40\alpha + 36\alpha = 238$, which yields $\alpha = 2$. Therefore, the steady state levels of the water in the reservoirs will be $\begin{bmatrix} 86 \\ 80 \\ 72 \end{bmatrix}$.

3. **Eigenvalues and Special Matrices – Visualization**

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$.

The following parts don't require knowledge about how to find eigenvalues. Answer each part by reasoning about the matrix at hand.

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Multiplying the identity matrix with any vector in $\mathbb{R}^n$ produces the same vector, that is, $\mathbf{I}\vec{x} = \vec{x} = 1 \cdot \vec{x}$. Therefore, $\lambda = 1$. Since $\vec{x}$ can be any vector in $\mathbb{R}^n$, the corresponding eigenvectors are all vectors in $\mathbb{R}^n$.

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Answer:

Since the matrix is diagonal, multiplying the diagonal matrix with any standard basis vector $\vec{e}_i$ produces $d_i\vec{e}_i$, that is, $\mathbf{D}\vec{e}_i = d_i\vec{e}_i$. Therefore, the eigenvalues are the diagonal entries d_i of $\mathbf{D}$, and the corresponding eigenvector associated with $\lambda = d_i$ is the standard basis vector $\vec{e}_i$.

(c) Does a rotation matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

There are three cases:

i. Rotation by 0° (more accurately, any integer multiple of 360°), which yields a rotation matrix $\mathbf{R} = \mathbf{I}$: This will have one eigenvalue of $+1$ because it doesn't affect any vector ($\mathbf{R}\vec{x} = \vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

ii. Rotation by 180° (more accurately, any angle of $180° + n \cdot 360°$ for integer n), which yields a rotation matrix $\mathbf{R} = -\mathbf{I}$: This will have one eigenvalue of -1 because it "flips" any vector ($\mathbf{R}\vec{x} = -\vec{x}$). The eigenspace associated with it is $\mathbb{R}^2$.

iii. Any other rotation: there aren't any real eigenvalues. The reason is, if there were any real eigenvalue $\lambda \in \mathbb{R}$ for a non-trivial rotation matrix, it means that we can get $\mathbf{R}\vec{x} = \lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$, which means that by rotating a vector, we scaled it. This is a contradiction (again, unless $\mathbf{R} = \mathbf{I}$). Refer to Figure 2 for a visualization.

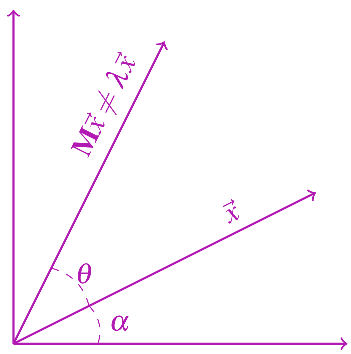

Figure 2: Rotation will never scale any non-zero vector (by a real number) unless it is rotation by an integer multiple of 360° (identity matrix) or the rotation angle is $\theta = 180° + n \cdot 360°$ for any integer n ($-\mathbf{I}$).

(d) Does a reflection matrix in $\mathbb{R}^2$ have any eigenvalues $\lambda \in \mathbb{R}$?

Answer:

Yes, both $+1$ and -1. Why? Reflecting any vector that is on the reflection axis will not affect it (eigenvalue $+1$). Reflecting any vector orthogonal to the reflection axis will just "flip it/negate it" (eigenvalue -1). In other words, the axis of reflection is the eigenspace associated with the eigenvalue $+1$ and the orthogonal space to that axis of reflection is the eigenspace associated with the eigenvalue -1. Refer to Figure 3 for a visualization.

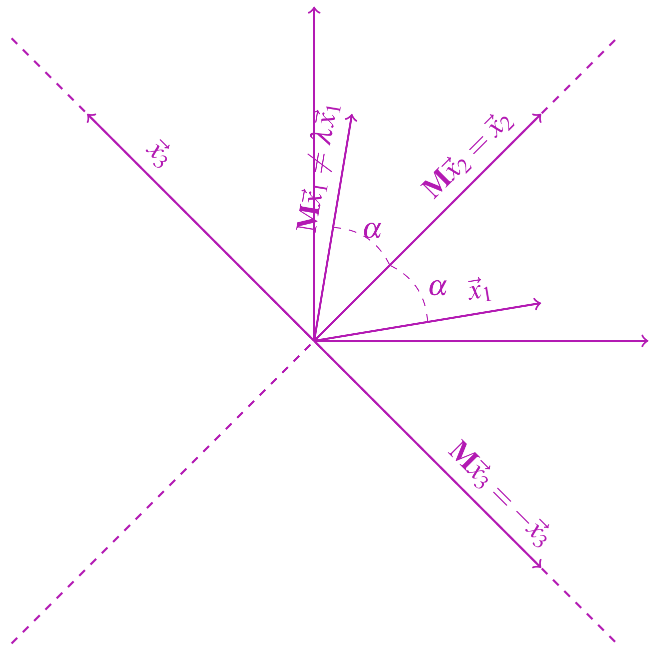

Figure 3: Reflection will scale vectors on the reflection axis (by $+1$) and orthogonal to it (by -1).

(e) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

Answer:

$\dim(\text{Null}(\mathbf{M})) > 0$

$\mathbf{M}\vec{x} = \vec{b}$ has no unique solution.

(f) Does the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

Hint: What is the rank of the matrix?

Answer:

Note that the matrix is rank-deficient. Therefore, according to part (e), one eigenvalue is $\lambda = 0$. The corresponding eigenvector, which is equivalent to the basis vector for the null space, is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The other eigenvalue is, by inspection, $\lambda = 1$ with the corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.