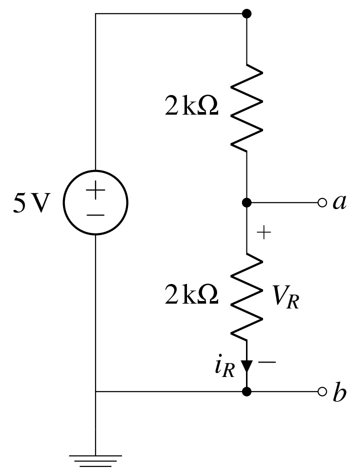
EECS 16A Spring 2019 — Designing Information Devices and Systems I — Discussion 8A

1. Why Bother With Thévenin Anyway?

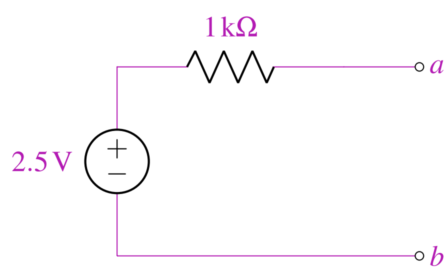
(a) Find a Thévenin eqiuvalent for the circuit shown below.



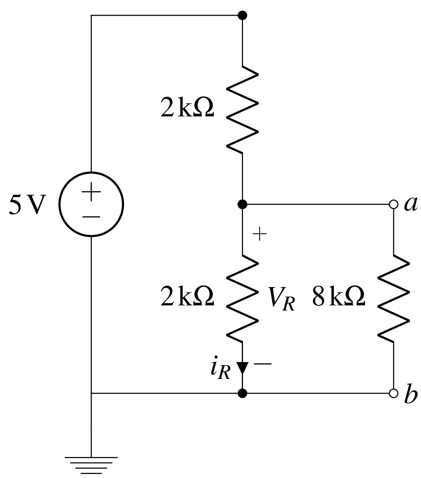
Answer:

$$V_{Th} = \frac{2\,\mathrm{k}\Omega}{2\,\mathrm{k}\Omega + 2\,\mathrm{k}\Omega} \cdot 5\,\mathrm{V} = 2.5\,\mathrm{V}$$

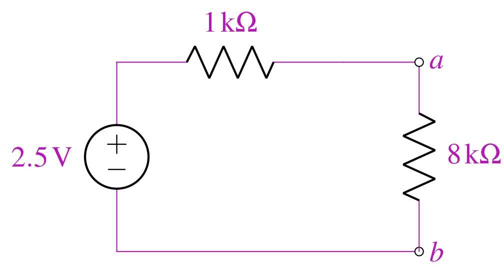
$$R_{Th} = 2\,\mathrm{k}\Omega \parallel 2\,\mathrm{k}\Omega = 1\,\mathrm{k}\Omega$$



(b) What happens to the output voltage V_{ab} if we attach a load of $8\,\mathrm{k}\Omega$ to the output as depicted in the circuit below? Use your Thévenin equivalent from part (a).



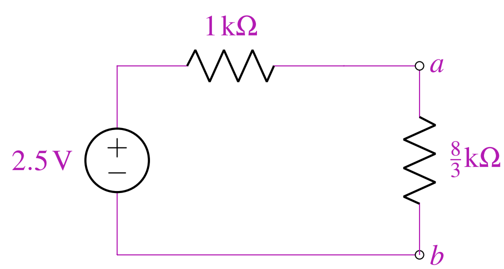
Answer:
We just attach the $8\,\text{k}\Omega$ resistor to our Thévenin equivalent circuit and calculate the voltage across it.



$$V_R = \frac{8\,\text{k}\Omega}{1\,\text{k}\Omega + 8\,\text{k}\Omega} \cdot 2.5\,\text{V} = 2.22\,\text{V}$$

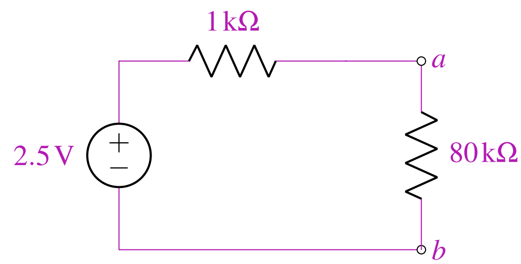
(c) What if the load is $\frac{8}{3}\text{k}\Omega$? What if the load is $80\,\text{k}\Omega$?

Answer:
$R = \frac{8}{3}\text{k}\Omega$:



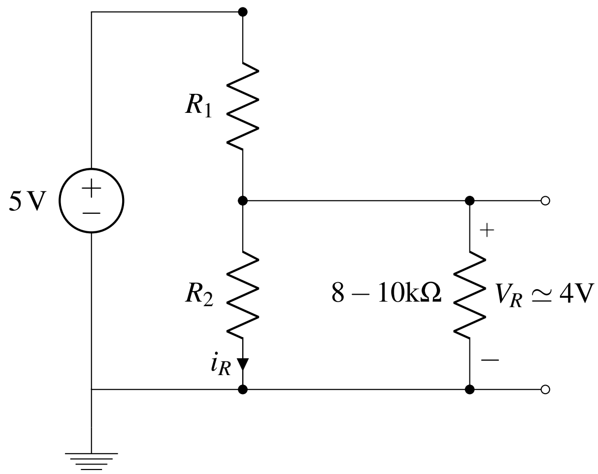
$$V_R = \frac{\frac{8}{3}\text{k}\Omega}{1\,\text{k}\Omega + \frac{8}{3}\text{k}\Omega} \cdot 2.5\,\text{V} = 1.82\,\text{V}$$

$R = 80\,\text{k}\Omega$:



$$V_R = \frac{80\,\text{k}\Omega}{1\,\text{k}\Omega + 80\,\text{k}\Omega} \cdot 2.5\,\text{V} = 2.46\,\text{V}$$

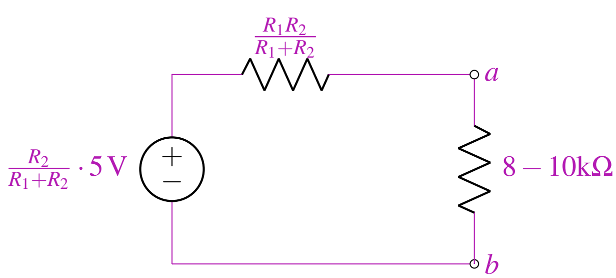
(d) Say that we want to support loads in the range of $8\,\text{k}\Omega$ to $10\,\text{k}\Omega$. We would like to maintain $4\,\text{V}$ across these loads. How can we approximately achieve this by setting R_1 and R_2 in the following circuit?



Answer:

$$V_{Th} = \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V}$$

$$R_{Th} = R_1 \parallel R_2 = \frac{R_1 R_2}{R_1 + R_2}$$



$$V_R = \frac{R}{R + \frac{R_1 R_2}{R_1 + R_2}} \cdot \frac{R_2}{R_1 + R_2} \cdot 5\,\text{V} \simeq 4\,\text{V}$$

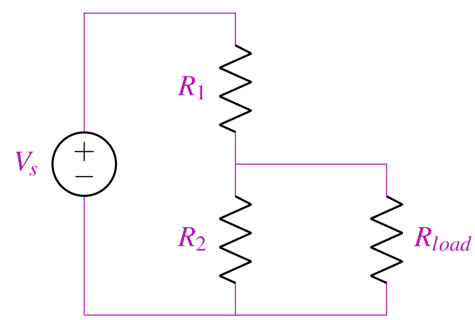
$$\frac{R R_2}{R(R_1 + R_2) + R_1 R_2} = \frac{4}{5}$$

If we set $R_1, R_2 \ll R$, then $\frac{RR_2}{R(R_1+R_2)+R_1R_2} \approx \frac{RR_2}{R(R_1+R_2)} = \frac{R_2}{R_1+R_2}$. Therefore, we can just choose two small resistors $R_1, R_2 \ll 8\,\mathrm{k}\Omega$, such that $R_2 = 4R_1$.

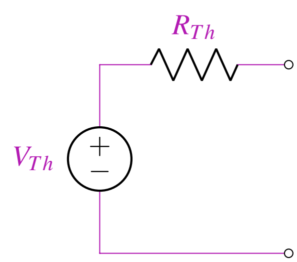
(e) For part (b), how much power does each element dissipate? Calculate the power using your Thévenin equivalent and using the original circuit. Are the values the same?

Answer:

We will ignore the power dissipated by R_{Th} initially and just explore V_s vs. V_{Th} and R_{load} in either case. This could be done for the specific example above, but it's more useful to go through this exercise generally. Thus, we will use the circuit shown below:



Recall that the Thévenin equivalent for the circuit above looks as follows:



where $R_{Th} = \frac{R_1R_2}{R_1+R_2}$ and $V_{Th} = \frac{R_2}{R_1+R_2}V_s$.

Because we are going to end up writing a few expressions multiple times, we are going to define a new variable:

$$\beta = R_1R_2 + R_{load}R_1 + R_{load}R_2$$

Let's start with our equivalent circuit. In the equivalent circuit, the current through the load resistor and equivalently every other element in the circuit is:

$$I = \frac{V_{ab}}{R_{load}} = \frac{V_{Th}}{R_{load} + R_{Th}}$$

With this current, we find the power dissapated across the source and the load resistor.

$$P_{V_{Th}} = -IV = -\frac{V_{Th}^2}{R_{load} + R_{Th}} = -\frac{V_{Th}^2(R_1+R_2)}{\beta} = -\frac{V_s^2R_2^2}{\beta(R_1+R_2)}$$

$$P_{R_{load}} = I^2R = \frac{V_{Th}^2}{(R_{load}+R_{Th})^2} \cdot R_{load} = \frac{V_{Th}^2(R_1+R_2)^2}{\beta^2} \cdot R_{load} = \frac{V_s^2R_2^2}{\beta^2} \cdot R_{load}$$

Let's try to find the answer from the original circuit. We will begin by calculating the current through the source.

$$I_s = \frac{V_s}{R_{eq}} = \frac{V_s}{R_1 + R_2 \parallel R_{load}} = \frac{V_s(R_1+R_2)}{\beta}$$

Now, we can calulate the power through the source.

$$P_{V_s} = -I_s V_s = -\frac{V_s^2(R_2+R_{load})}{\beta}$$

The power dissipated by the source in the original circuit is not the same as the power dissipated in the new circuit. What about the load resistor? We will first calculate the voltage across the load resistor.

$$V_{load} = \frac{R_2 \parallel R_{load}}{R_1 + R_2 \parallel R_{load}} \cdot V_s = \frac{\frac{R_2 R_{load}}{R_2+R_{load}}}{R_1 + \frac{R_2 R_{load}}{R_2+R_{load}}} \cdot V_s = \frac{R_2 R_{load}}{\beta} \cdot V_s$$

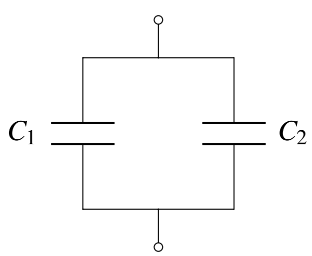
$$P_{load} = \frac{V_{load}^2}{R_{load}} = \frac{V_s^2 R_2^2}{\beta^2} R_{load}$$

The power through the load is the same! Thévenin equivalents can be used to calculate the power through elements that are not part of the circuit that was transformed.

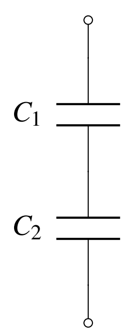
2. Series And Parallel Capacitors

Derive C_{eq} for the following circuits.

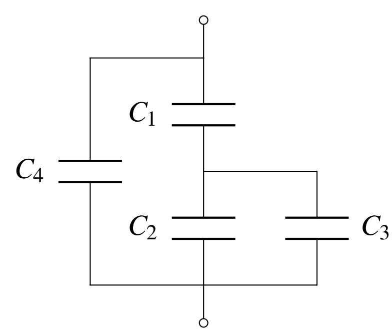
(a)



(b)



(c)

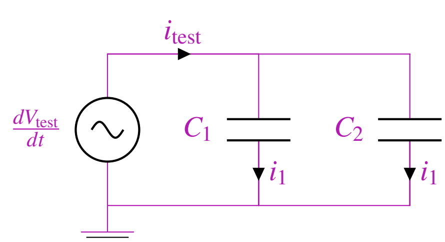


Answer:

(a)

$$C_{eq} = C_1 + C_2$$

Notice these capacitors are in parallel. We can derive their equivalent capacitance by connecting them to a voltage source with a constant derivative, as shown by the circuit below:



Since both capacitors have the same voltage across them:

$$\frac{dV_{C_1}}{dt} = \frac{dV_{C_2}}{dt} = \frac{dV_{\text{test}}}{dt}$$

$$i_1 = C_1 \frac{dV_{\text{test}}}{dt}$$

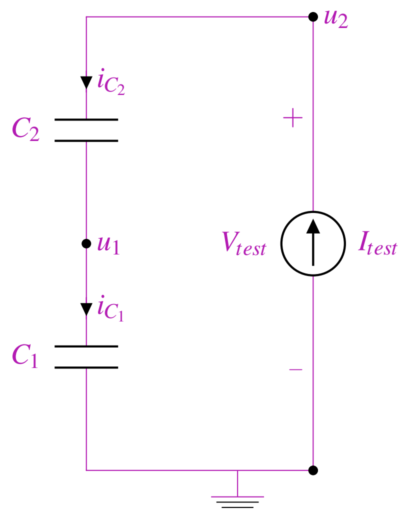
$$i_2 = C_2 \frac{dV_{\text{test}}}{dt}$$

$$i_t = i_1 + i_2 = (C_1 + C_2)\frac{dV_{\text{test}}}{dt}$$

Since we know $i_{\text{test}} = C_{\text{eq}} \frac{dV_{\text{out}}}{dt}$,

$$C_{\text{eq}} = C_1 + C_2$$

(b) In order to find the equivalence capacitance of the circuit, we plug in a test current source, and measure the rate of change of voltage across it.



From KCL, we know that all of the currents are equal.

$$i_{C_1} = i_{C_2} = I_{test}$$

For each capacitor, we plug in our $I - \frac{dV}{dt}$ relationship:

$$i_{C_1} = I_{test} = C_1 \frac{du_1}{dt}$$

$$i_{C_2} = I_{test} = C_2 \frac{d(u_2 - u_1)}{dt} = C_2 \left(\frac{du_2}{dt} - \frac{du_1}{dt} \right)$$

Next, we eliminate u_1 from the equations above and rearrange.

$$\frac{du_1}{dt} = \frac{I_{test}}{C_1} \Rightarrow I_{test} = C_2 \frac{du_2}{dt} - \frac{C_2}{C_1} I_{test}$$

$$I_{test} = \frac{C_2}{1 + \frac{C_2}{C_1}} \frac{du_2}{dt}$$

Finally, we plug in that $u_2 = V_{test}$ and solve for the equivalent capacitance with $C_{eq} = I_{test} / \frac{dV_{test}}{dt}$

$$I_{test} = \frac{C_2}{1+\frac{C_2}{C_1}} \frac{dV_{test}}{dt}$$

$$\Rightarrow C_{eq} = \frac{C_2}{1+\frac{C_2}{C_1}} = \frac{C_1 C_2}{C_1 + C_2}$$

Note that this is the same as saying $C_{eq} = C_1 \parallel C_2$. Remember that the $\parallel$ operator is mathematical notation; in this case, the capacitors are actually in series, but *mathematically* their equivalent circuit is found via the "parallel resistor" operation.

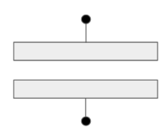
(c) Given that we know what the relationship for capacitors in series and parallel are from the last two parts, we can just simply the capacitors step by step:

$$C_{eq} = (C_4 + (C_1 \parallel (C_2 + C_3))) = \frac{C_4(C_1 + C_2 + C_3) + C_1(C_2 + C_3)}{C_1 + C_2 + C_3}$$

3. Capacitance Equivalence

For the structures shown below, assume that the plates have a depth L into the page and a width W and are always a distance d apart.

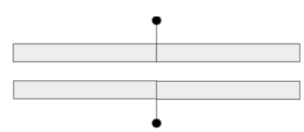
(a) What is the capactance of the structure shown below?



Answer:

The capacitance of two parallel plate conductors is given by $C = \varepsilon \frac{A}{d}$. The cross-sectional area A is WL, so the capacitance is $C = \varepsilon \frac{WL}{d}$.

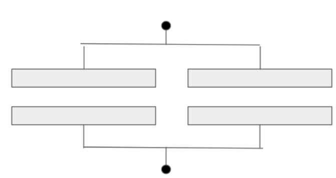
(b) Suppose that we take two such structures and put them next to each other as shown below. What is the capacitance of this new structure?



Answer:

Here, we have just doubled the width of the capacitor plates. The new capacitance is $C = \varepsilon \frac{2WL}{d}$. Notice that this is just double the capacitance from the first part.

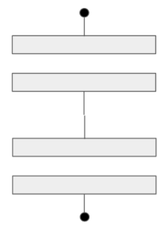
(c) Now suppose that rather than connecting the together as shown above, we connect them with an ideal wire as shown below. What is the capacitance of this structure?



Answer:

Intuitively, nothing has changed here since we have just added an ideal wire between two capacitors. Thus, the answer remains $C = \varepsilon \frac{2WL}{d}$.

(d) Suppose that we now take two capacitors and connect them as shown below. What is the capacitance of the structure?

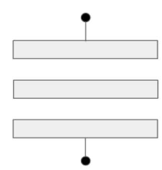


Answer:

We know that capacitors placed in series follow the parallel rule. Thus, the overall capacitance is half the individual capacitance.

$$C_{eq} = C||C = \frac{C \cdot C}{C + C} = \frac{C}{2}$$

(e) What is the capacitance of the structure shown below?



Answer:

Notice here that we are ignoring the material in the middle. Thus, from a modeling perspective, we can think of this as the original capacitor with the distance between the plates doubled.

$$C_{eq} = \varepsilon \frac{WL}{2d} = \frac{1}{2} \varepsilon \frac{WL}{2d} = \frac{C}{2}$$

