EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 12A

1. Mechanical Projection

In $\mathbb{R}^n$, the projection of vector $\vec{a}$ onto vector $\vec{b}$ is defined as:

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\| \vec{b} \right\|^2} \vec{b} = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\| \vec{b} \right\|} \hat{b}$$

where $\hat{b}$ is the normalized $\vec{b}$, i.e., a unit vector with the same direction as $\vec{b}$.

(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ – that is, onto the x-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ – that is, onto the y-axis. Graph these two vectors and the projection.

(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

(e) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ – that is, onto the x-y plane in $\mathbb{R}^3$.

(f) What is the geometric/physical interpretation of projection? Justify using the previous parts.

(g) For part $(a)(b)$ and part $(c)(d)$, we looked at two different projections for one vector. For those cases, using only the projected vector $\text{proj}_{\vec{b}}(\vec{a})$ and the vector $\vec{b}$ we projected onto, do we have enough information to reconstruct the original vector $\vec{a}$?

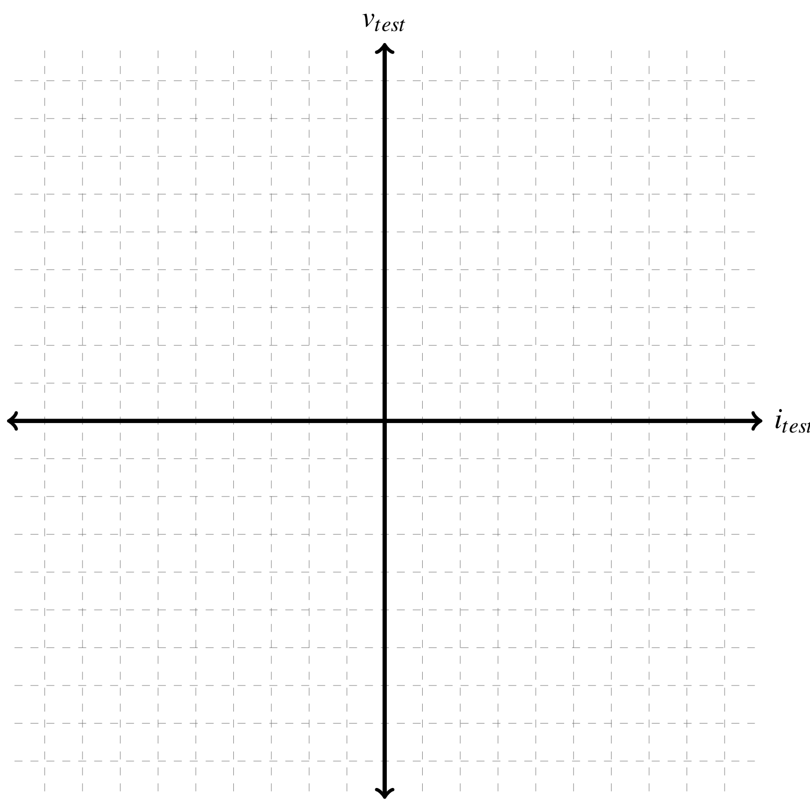
(h) Given information about n projections of a vector in $\mathbb{R}^n$, when do we have enough information to reconstruct the original vector?

2. Ohm's Law With Noise

We are trying to measure the resistance of a black box. We apply various i_{test} currents and measure the ouput voltage v_{test}. Sometimes, we are quite fortunate to get nice numbers. Oftentimes, our measurement tools are a little bit noisy, and the values we get out of them are not accurate. However, if the noise is completely random, then the effect of it can be averaged out over many samples. So we repeat our test many times:

Test	i_{test} (mA)	v_{test} (V)
1	10	21
2	3	7
3	-1	-2
4	5	8
5	-8	-15
6	-5	-11

(a) Plot the measured voltage as a function of the current.



(b) Suppose we stack the currents and voltages to get $\vec{I} = \begin{bmatrix} 10 \\ 3 \\ -1 \\ 5 \\ -8 \\ -5 \end{bmatrix}$ and $\vec{V} = \begin{bmatrix} 21 \\ 7 \\ -2 \\ 8 \\ -15 \\ -11 \end{bmatrix}$. Is there a unique solution for R? What conditions must $\vec{I}$ and $\vec{V}$ satisfy in order for us to solve for R uniquely?

(c) Ideally, we would like to find R such that $\vec{V} = \vec{I}R$. If we cannot do this, we'd like to find a value of R that is the *best* solution possible, in the sense that $\vec{I}R$ is as "close" to $\vec{V}$ as possible. We are defining the sum of squared errors as a **cost function**. In this case the cost function for any value of R quantifies the difference between each component of $\vec{V}$ (i.e. v_j) and each component of $\vec{I}R$ (i.e. i_jR) and sum up the squares of these "differences" as follows:

$$\text{cost}(R) = \sum_{j=1}^{6} (v_j - i_jR)^2$$

Do you think this is a good cost function? Why or why not?

(d) Show that you can also express the above cost function in vector form, that is,

$$\text{cost}(R) = \left\langle (\vec{V} - \vec{I}R), (\vec{V} - \vec{I}R) \right\rangle$$

Hint: $\langle \vec{a}, \vec{b} \rangle = \vec{a}^T \vec{b} = \sum_i a_i b_i$

(e) Find $\hat{R}$, which is defined as the optimal value of R that minimizes $\text{cost}(R)$.

Hint: Use calculus. The optimal $\hat{R}$ makes $\frac{d\text{cost}(\hat{R})}{dR} = 0$

(f) On your original IV plot, also plot the line $v_{test} = \hat{R}i_{test}$. Can you visually see why this line "fits" the data well? How well would we have done if we had guessed $R = 3\,\text{k}\Omega$? What about $R = 1\,\text{k}\Omega$? Calculate the cost functions for each of these choices of R to validate your answer.

(g) Now, suppose that we add a new data point: $i_7 = 2\,\text{mA}$, $v_7 = 4\,\text{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R}i_{test}$?

(h) Let's add another data point: $i_8 = 4\,\text{mA}, v_8 = 11\,\text{V}$. Will $\hat{R}$ increase, decrease, or remain the same? Why? What does that say about the line $v_{test} = \hat{R}i_{test}$?

(i) Now your mischievous friend has hidden the black box. You want to predict what output voltage across the terminals if you applied 5.5 mA through the black box. What would your best guess be?

