EECS 16A Designing Information Devices and Systems I
Spring 2019 Discussion 13B

In Fall 2018, this discussion was not held haha

1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:

- A set of m songs, each of length n: $\mathbf{S} = \{\vec{S}_0, \vec{S}_1, ..., \vec{S}_{m-1}\}$
- An n-dimensional received signal vector: $\vec{r}$
- The sparsity level k of the signal
- Some threshold, th. When the norm of the signal is below this value, the signal contains only noise.

Outputs:

- A set of songs that were identified, F, which will contain at most k elements
- A vector $\vec{x}$ containing song messages $(x_1, x_2, \ldots)$, which will be of length k or less
- An n-dimensional residual $\vec{y}$

Procedure:

- Initialize the following values: $\vec{y} = \vec{r}$, $j = 1$, k, $\mathbf{A} = \begin{bmatrix} & \end{bmatrix}$, $F = \{\emptyset\}$
- while $((j \leq k)$ and $(\| \vec{y} \| \geq th))$:
  (a) Cross-correlate $\vec{y}$ with the shifted versions of all songs. Find the song index i and the shifted version of the song, $\vec{S}_i^N$, with which the received signal has the highest correlation value.
  (b) Add i to the set of song indices F.
  (c) Column concatenate matrix $\mathbf{A}$ with the correctly shifted version of the song: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{S}_i^N \end{bmatrix}$
  (d) Use least squares to obtain the message value: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{r}$
  (e) Update the residual value $\vec{y}$ by subtracting: $\vec{y} = \vec{r} - \mathbf{A}\vec{x}$
  (f) Update the counter: $j = j + 1$

2. Orthogonal Matching Pursuit

Let's work through an example of the OMP algorithm. Suppose that we have a vector $\vec{x} \in \mathbb{R}^4$ that is sparse and we know that it has only 2 non-zero entries. In particular,

$$\mathbf{M}\vec{x} \approx \vec{y} \tag{1}$$

$$\begin{bmatrix} | & | & | & | \\ \vec{m}_1 & \vec{m}_2 & \vec{m}_3 & \vec{m}_4 \\ | & | & | & | \end{bmatrix} \vec{x} \approx \vec{y} \tag{2}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \tag{3}$$

where exactly 2 of x_1 to x_4 are non-zero. Use Orthogonal Matching Pursuit to estimate x_1 to x_4.

(a) Why can we not solve for $\vec{x}$ directly?

(b) Why can we not apply the least squares process to obtain $\vec{x}$?

(c) Let us start by reviewing the OMP procedure,

Inputs:

- A matrix $\mathbf{M}$, whose columns, $\vec{m}_i$, make up a set of vectors, $\{\vec{m}_i\}$, each of length n
- A vector $\vec{y}$ of length n
- The sparsity level k of the signal

Outputs:

- A vector $\vec{x}$, that contains k non-zero entries.
- A error vector $\vec{e} = \vec{y} - \mathbf{M}\vec{x}$

Procedure:

- Initialize the following values: $\vec{e} = \vec{y}$, $j = 1$, k, $\mathbf{A} = \begin{bmatrix} & \end{bmatrix}$
- while ($j \leq k$):
  i. Compute the inner product for each vector in the set, $\vec{m}_i$, with $\vec{e}$: $c_i = \langle \vec{m}_i, \vec{e} \rangle$.
  ii. Column concatenate matrix $\mathbf{A}$ with the column vector that had the maximum inner product value with $\vec{e}$, c_i: $\mathbf{A} = \begin{bmatrix} \mathbf{A} & | & \vec{m}_i \end{bmatrix}$
  iii. Use least squares to compute $\vec{x}$ given the $\mathbf{A}$ for this iteration: $\vec{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$
  iv. Update the error vector: $\vec{e} = \vec{y} - \mathbf{A}\vec{x}$
  v. Update the counter: $j = j + 1$

(d) Compute the inner product of every column with the $\vec{y}$ vector. Which column has the largest inner product? This will be the first column of the matrix $\mathbf{A}$.

(e) Now, find the projection of $\vec{y}$ onto the columns of $\mathbf{A}$ (ie. $\text{proj}_{\text{Col}(\mathbf{A})}\vec{y} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{y}$). Use this to update the error vector.

(f) Now compute the inner product of every column with the new error vector. Which column has the largest inner product? This will be the second column of $\mathbf{A}$.

(g) We now have two non-zero entries for our vector, $\vec{x}$. Find the values of those two entries.

(Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)