

1. Constructing a Basis

Let's consider a subspace of \mathbb{R}^3 called V which has the following property: for every vector in V , the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V . What is the dimension of V ?

2. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space \mathbb{R}^m and a set of n vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ in \mathbb{R}^m .

- (a) For the first part of the problem, let $m > n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? Why/why not? What conditions would we need?
- (b) Let $m = n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? Why/why not? What conditions would we need?
- (c) Now, let $m < n$. Can $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ form a basis for \mathbb{R}^m ? What vector space could they form a basis for?

Hint: Think about whether the vectors can be linearly independent.

3. Exploring Column Spaces and Null Spaces

- The **column space** is the possible outputs of a transformation/function/linear operation. It is also the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

- i. What is the column space of \mathbf{A} ? What is its dimension?
- ii. What is the null space of \mathbf{A} ? What is its dimension?
- iii. Are the column spaces of the row reduced matrix \mathbf{A} and the original matrix \mathbf{A} the same?
- iv. Do the columns of \mathbf{A} form a basis for \mathbb{R}^2 (or \mathbb{R}^3 for (e))? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$$(d) \begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$$

$$(e) \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$