
EECS 16A Designing Information Devices and Systems I Homework 14
 Spring 2019

You should plan to complete this homework by Thursday, May 10th. Everything in this homework is in scope for the final, but you do not need to turn anything in. There are no self-grades for this homework.

1. Mechanical Gram-Schmidt

Use Gram-Schmidt to find a matrix \mathbf{U} whose columns form an orthonormal basis for the column space of \mathbf{V} .

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that you get the same resulting vector when you project $\vec{w} = [1 \ -1 \ 0 \ -1 \ 0]^T$ onto \mathbf{V} and onto \mathbf{U} , i.e. show that

$$\mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\vec{w} = \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T\vec{w}.$$

2. Speeding Up OMP

Consider the Sparse Imaging problem in homework 13.

(a) Modify the code to run faster by using a Gram-Schmidt orthonormalization to speed it up. (Edit the code given to you in `prob14.ipynb`.)

(b) **PRACTICE:** Do any other modifications you want to further speed up the code.

Hint: When possible, how would you safely extract multiple peaks corresponding to multiple pixels in one go and add them to the recovered list? Would this speed things up?

3. Mechanical: Change of Basis

All calculations in this problem are intended to be done by hand, but you can use a computer to check your work.

(a) Consider two bases for \mathbb{R}^2 , A and B , represented by the columns of matrices \mathbf{A} and \mathbf{B} , respectively.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Suppose \vec{x}_A represents the coordinates of a vector \vec{x} in basis A and \vec{x}_B represents the coordinates of the same vector in basis B . Write the coordinate transformation that converts \vec{x}_A to \vec{x}_B . That is, find the matrix \mathbf{T} , such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A.$$

- (b) Consider two bases for \mathbb{R}^2 , A and B , represented by the columns of matrices \mathbf{A} and \mathbf{B} , respectively.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Suppose \vec{x}_A represents the coordinates of a vector \vec{x} in basis A and \vec{x}_B represents the coordinates of the same vector in basis B . Write the coordinate transformation that converts \vec{x}_A to \vec{x}_B , that is, find the matrix \mathbf{T} , such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A.$$

- (c) Consider two bases for \mathbb{R}^3 , A and B , represented by the columns of matrices \mathbf{A} and \mathbf{B} , respectively.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Suppose \vec{x}_A represents the coordinates of a vector \vec{x} in basis A and \vec{x}_B represents the coordinates of the same vector in basis B . Write the coordinate transformation that converts \vec{x}_A to \vec{x}_B , that is, find the matrix \mathbf{T} , such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A$$

Hint: What do you notice about A and B that will simplify this calculation?

4. Completely Normal Eigenvectors

- (a) Consider matrix A that has eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$A = \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0.5 & 2.5 & -0.5 \\ 0. & 0. & 4. \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormalize the eigenvectors using Gram-Schmidt to get vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Perform the orthonormalization in the order $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

- (b) Write the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as a linear combination of the eigenvectors. Are any of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ still eigenvectors of the matrix A ? Justify your answer.
- (c) Let $U = [\vec{u}_1 \ \vec{u}_2 \ \vec{u}_3]$. Calculate $U^T \cdot U$.
- (d) Prove that if an arbitrary matrix X has orthogonal eigenvectors then X is symmetric i.e. $X^T = X$. You may assume that X exists in $\mathbb{R}^{n \times n}$ and has n linearly independent eigenvectors.