EECS 16A Designing Information Devices and Systems I
Spring 2019 Homework 4

This homework is due February 22, 2019, at 23:59.
Self-grades are due February 26, 2019, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

  If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. Finding Null Spaces

(a) Consider the column vectors of any 3×5 matrix. What is the maximum possible number of linearly independent column vectors?

(b) Someone performed Gaussian elimination and got the following upper triangular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of $\mathbf{A}$. How many unique vectors are required to span the column space of $\mathbf{A}$? (This is the dimension of the column space of $\mathbf{A}$)

(c) Recall that for every vector $\vec{x}$ in the null space of $\mathbf{A}$, we have $\mathbf{A}\vec{x} = \vec{0}$. The dimension of a the null space is the minimum number of vectors needed to span it. Find vectors that span the null space of $\mathbf{A}$ (the matrix in the previous part). What is the dimension of the null space of $\mathbf{A}$? Use the same $\mathbf{A}$ from part b.

(d) (**Practice**) Now consider the new matrix, B, which is related to A,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of $\mathbf{B}$. How many unique vectors are required to span the column space of $\mathbf{B}$?

(e) Find vector(s) that span the null space of the following matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

2. Mechanical Eigenvalues and Eigenvectors

Find the eignvalues and their eigenspaces — give a basis for the eigenspace when it is more than 1 dimensional

(a) $\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

(c) **(PRACTICE)** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)

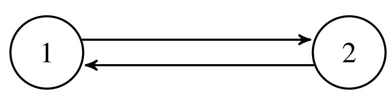
(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

3. Counting The Paths of a Random Surfer

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from one webpage to another webpage. To do this, we represent the webpages as a graph.

If webpage 1 has a link to webpage 2, we have a directed edge from webpage 1 to webpage 2. This graph can further be represented by what is known as an "adjacency matrix", $\mathbf{A}$, with elements a_{ij}. We define $a_{ji} = 1$ if there is link from page i to page j. Note the ordering of the indices! Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j.

Consider the following graphs.



Graph A

(a) Based on this definition, the "adjacency matrix" for graph A, will be,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1}$$

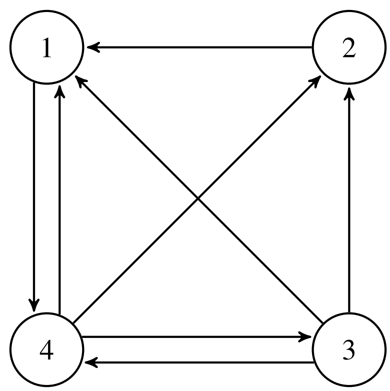
The element a_{ji} of $\mathbf{A}$ gives the number of one-hop paths from from website i to website j. Similarly, the elements of $\mathbf{A}^2$ give the number of two-hop paths from website i to website j. How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

(b) This path counting aspect is very related to the steady-state frequency for the fraction of people for each webpage. The steady-state frequency for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the "transition matrix" of the graph.

The "transition matrix", $\mathbf{T}$, is slightly different from the "adjacency matrix". Its values, t_{ji}, are the *proportion* of the people who are at website i that click the link for website j. We assume people divide equally among the links on the website (e.g. if there are three links on a website, $\frac{1}{3}$ of the people will click each link).

Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When this eigenvector's values are made to sum to one (to conserve people), the i^{th} element of the eigenvector corresponds to the fraction of people on the i^{th} website.

For graph A, what are the steady-state frequencies for the two webpages?

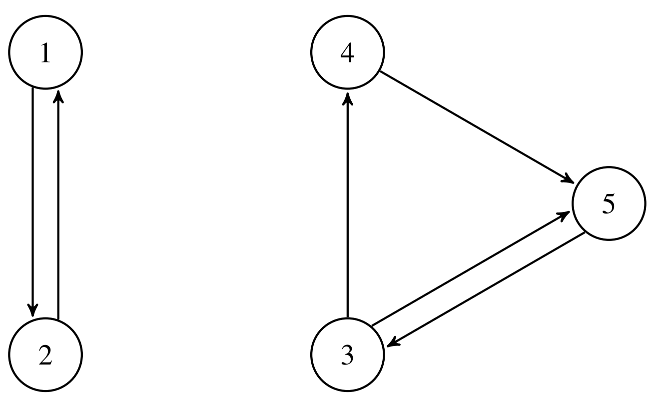


Graph B

(c) Write out the adjacency matrix for graph B.

(d) For graph B, how many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2? You may use IPython for this.

(e) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command numpy.linalg.eig for this.



Graph C

(f) Write out the adjacency matrix for graph C.

(g) For graph C, how many paths are there from webpage 1 to webpage 3?

(h) **(PRACTICE)** Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C?

4. Traffic Flows

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this "flow conservation" to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem, we will explore this concept.

(a) Let's begin with a network with three intersections, A, B and C. Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A, flow t_2 as the rate on the road between C and B, and flow t_3 as the rate on the road between C and A.

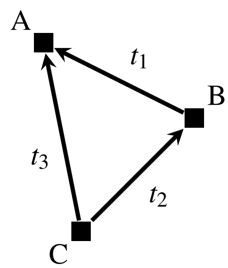


Figure 1: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C, then $t_3 = -100$.)

We assume the "flow conservation" constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B, we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3.

(b) Now suppose we have a larger network, as shown in Figure 2.

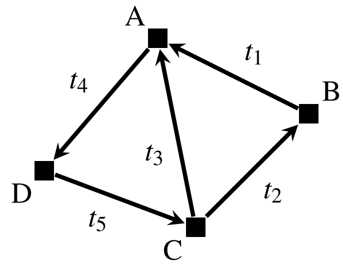


Figure 2: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads AD (measuring t_4) and BA (measuring t_1). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is it possible to determine all traffic flows, $\begin{bmatrix} t_1, t_2, t_3, t_4, t_5 \end{bmatrix}^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

(c) We would like a more general way of determining the possible traffic flows in a network. Suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow conservation constraints (one per intersection) as a matrix equation.

Construct a 4×5 matrix $\mathbf{B}$ such that the equation $\mathbf{B}\vec{t} = \vec{0}$

$$\begin{bmatrix} & & \mathbf{B} & & \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network in Figure 2.

Hint: Each row is the constraint of an intersection. You can construct $\mathbf{B}$ using only $0, 1,$ and -1 entries. This matrix is called the **incidence matrix.** What constraint does each column of $\mathbf{B}$ represent?

(d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace of all valid traffic flows for the network of Figure 2. Specifically, express this space as the span of two linearly independent vectors.

Hint: Use the claim of the Berkeley student in part (b). Justify why you can use their claim. Then write all valid flows as a vector in terms of $t_1 = \alpha$ and $t_4 = \beta$.

(e) Notice that the set of all vectors $\vec{t}$ that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix $\mathbf{B}$. That is, we can find all valid traffic flows by computing the null space of $\mathbf{B}$. Use Gaussian elimination to determine the dimension of the null space of $\mathbf{B}$ and compute a basis for the null space. Does this match your answer to part (d)?

Challenge (optional): Can you interpret the dimension of the null space of $\mathbf{B}$ for the road networks of Figure 1 and Figure 2?

(f) Now let us analyze more general road networks. Say there is a road network graph G, with incidence matrix $\mathbf{B}_G$. If $\mathbf{B}_G$ has a k-dimensional null space, does this mean measuring the flows along ***any*** k **roads** is always sufficient to recover the exact flows? Prove or give a counterexample.

Hint: Consider the Stanford student from part (b).

(g) Let G be a network of n roads with the incidence matrix $\mathbf{B}_G$, which has a k-dimensional null space. We would like to characterize exactly when it is sufficent to measure a set of k roads to recover the exact flow along all roads.

To do this, it will help to generalize the problem and consider measuring *linear combinations* of flows. Let t_i be the flow on one road. We measure some linear combination of t_i's or $m_0 \cdot t_0 + m_1 \cdot t_1 + \cdots + m_n \cdot t_n$. Now we measure many of these linear combinations, which we will represent using matrix vector multiplication. Then, making k measurements is equivalent to observing the vector $\mathbf{M}\vec{t}$ for some $k \times n$ "measurement matrix" $\mathbf{M}$.

For example, for the network of Figure 2, the measurement matrix corresponding to measuring t_1 and t_4 (as the Berkeley student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, the measurement matrix corresponding to measuring t_1 and t_2 (as the Stanford student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For general networks G and measurements $\mathbf{M}$, give a condition for when the exact traffic flows can be recovered in terms of the null space of $\mathbf{M}$ and the null space of $\mathbf{B}_G$.

Hint: Recovery will fail iff (if and only if) there are two valid flows with the same measurements, that is, there exist distinct $\vec{t}_1$ and $\vec{t}_2$ satisfying the flow conservation constraints, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$. Can you express this in terms of the null spaces of $\mathbf{M}$ and $\mathbf{B}_G$?

(h) **Challenge (optional):** If the incidence matrix $\mathbf{B}_G$ has a k-dimensional null space, does this mean we can **always pick a set of k roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

5. (PRACTICE) Codes Revisited

Alice and Bob are back and they've successfully figured out how to avoid dropping symbols when sending messages. In this problem, Alice is using a similar encoding scheme as last time where she uses vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to encode her message $\begin{bmatrix} a & b & c \end{bmatrix}^T$. (Assume Bob knows the vectors, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, that Alice is using.) Namely, she tries to send $\vec{k}$:

$$\vec{k} = \begin{bmatrix} - & \vec{v}_1^{\,T} & - \\ - & \vec{v}_2^{\,T} & - \\ - & \vec{v}_3^{\,T} & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{2}$$

Unfortunately, their arch-nemesis, Eve, is trying to interfere with Alice's messages to Bob and has found a way to add noise to the transmission! But, Eve's interference must pass through a linear transformation, $\mathbf{U}$, before the interference hits Alice's message. Now instead of seeing $\vec{k}$, Bob is receiving $\vec{y}$:

$$\vec{y} = \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ - & \vec{v}_3^T & - \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \tag{3}$$

where Eve inserts $\begin{bmatrix} p & q & r \end{bmatrix}^T$ and it undergoes a transformation by the matrix **U** and then undergoes the same transformation as Alice's original 3-symbol vector. There are two ways Eve's meddling can mess up the transmission:

- If Bob receives $\vec{0}$, he doesn't even realize he's getting a message
- Bob receives a nonzero transmission but can't determine the original $\begin{bmatrix} a & b & c \end{bmatrix}^T$

(a) **(PRACTICE)** Alice is using the following vectors for her encoding scheme

$$\vec{v}_1^T = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{v}_2^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

If Eve is not interfering ($p = q = r = 0$), will Bob be able to uniquely determine what a, b, and c are?

(b) **(PRACTICE)** Eve decides to change her strategy—now she's sending her interference through its own transformation before adding it to Alice's message. Bob receives $\vec{y}$ according to Equation (3).

Eve's interference is transformed by the vectors:

$$\vec{u}_1^T = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{u}_2^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \tag{4}$$

Find the null space of **U** = $\begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix}$

(c) **(PRACTICE)** Given **U** from Equation (4), find p, q, and r such that Bob is guaranteed to never realize he's receiving a message, regardless of Alice's encoding scheme, i.e. $\vec{y} = \vec{0}$ (this corresponds to the fact that the null space of any matrix always includes the zero vector).

State any constraints on a, b, and c which are necessary for Eve's cancellation to work. In other words, find $\begin{bmatrix} p & q & r \end{bmatrix}^T$ and define any restrictions on a, b, and c such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \vec{0}$$

Do not take into account the vectors Alice is using for her encoding.

(d) **(PRACTICE)** Realizing her scheme from (a) was flawed, Alice has also chosen to change things up and is using new vectors for her encoding:

$$\vec{v}_1^T = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \vec{v}_2^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 & 3 & 1 \end{bmatrix}$$

For each of the following cases, determine whether Bob will receive a message at all and—if he does—whether he'll be able to correctly and uniquely determine what a, b, and c are. Justify your answer. **Do not use IPython to solve this problem**. *Hint: Use your answers from (b) and (c)*

i. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

ii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

iii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

