EECS 16A Designing Information Devices and Systems I
Spring 2019 Homework 14

You should plan to complete this homework by Thursday, May 10th. Everything in this homework is in scope for the final, but you do not need to turn anything in. There are no self-grades for this homework.

1. Mechanical Gram-Schmidt

Use Gram-Schmidt to find a matrix $\mathbf{U}$ whose columns form an orthonormal basis for the column space of $\mathbf{V}$.

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that you get the same resulting vector when you project $\vec{w} = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \end{bmatrix}^T$ onto $\mathbf{V}$ and onto $\mathbf{U}$, i.e. show that

$$\mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\vec{w} = \mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T\vec{w}.$$

Solution:

We start with the columns of $\mathbf{V}$ as our basis for the column space of $\mathbf{V}$, and we want to find an orthonormal basis for this same space using Gram-Schmidt. For notational convenience, define

$$\mathbf{V} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$$

We summarize the first few steps of the Gram-Schmidt algorithm as follows:

(a) $\vec{u}_1' = \vec{v}_1$; $\quad \vec{u}_1 = \frac{\vec{u}_1'}{\|\vec{u}_1'\|}$.

(b) $\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1$; $\quad \vec{u}_2 = \frac{\vec{u}_2'}{\|\vec{u}_2'\|}$.

(c) $\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2$; $\quad \vec{u}_3 = \frac{\vec{u}_3'}{\|\vec{u}_3'\|}$.

For the column space of $\mathbf{V}$, this is

(a) $\vec{u}_1' = \vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Since $\vec{u}_1'$ is already normalized, we simply set $\vec{u}_1 = \vec{u}_1'$.

(b)

$$\vec{u}_2' = \vec{v}_2 - \langle \vec{v}_2, \vec{u}_1 \rangle \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$$

UCB EECS 16A, Spring 2019, Homework 14, All Rights Reserved. This may not be publicly shared without explicit permission.

(c)

$$\vec{u}_3' = \vec{v}_3 - \langle \vec{v}_3, \vec{u}_1 \rangle \vec{u}_1 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{\sqrt{2}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus, the matrix $\mathbf{U}$ is given by

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that when we have two different bases for a subspace, the two projections are still the same. So when we project the vector $\vec{w} = \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \end{bmatrix}^T$ onto the subspace using the $\mathbf{V}$ basis, we get

$$\mathbf{V}(\mathbf{V}^T\mathbf{V})^{-1}\mathbf{V}^T\vec{w} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\mathbf{U}(\mathbf{U}^T\mathbf{U})^{-1}\mathbf{U}^T\vec{w} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

All Rights Reserved. This may not be publicly shared without explicit permission.

Note however that the projection using the $\mathbf{U}$ basis was much simpler. Since $\mathbf{U}^T\mathbf{U}$ is the identity, we didn't need to do a matrix inversion.

2. Speeding Up OMP

Consider the Sparse Imaging problem in homework 13.

(a) Modify the code to run faster by using a Gram-Schmidt orthonormalization to speed it up. (Edit the code given to you in `prob14.ipynb`.)

Solution:

See `sol14.ipynb`.

The solution in the IPython notebook does Gram-Schmidt as it goes along and finds vectors. (The code is far from efficient in its implementation since it copies vectors too often.)

(b) **PRACTICE:** Do any other modifications you want to further speed up the code.

Hint: When possible, how would you safely extract multiple peaks corresponding to multiple pixels in one go and add them to the recovered list? Would this speed things up?

Solution:

See `sol14.ipynb`.

This approach grabs many pixels at once. This saves a lot of effort in computing correlations over and over again. The threshold to decide how many pixels are safe to grab is chosen based on knowledge that derives from topics covered in 126 and touched upon in 70. Play with the number 6. For example, try to change it to 1 or 2. See what happens. Notice that it starts grabbing some false pixels.

3. Mechanical: Change of Basis

All calculations in this problem are intended to be done by hand, but you can use a computer to check your work.

(a) Consider two bases for $\mathbb{R}^2$, A and B, represented by the columns of matrices $\mathbf{A}$ and $\mathbf{B}$, respectively.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$. That is, find the matrix $\mathbf{T}$, such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A.$$

Solution:

For a given vector $\vec{x}$, we have that

$$\vec{x} = \mathbf{A}\vec{x}_A = \mathbf{B}\vec{x}_B$$

Thus, the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{B}^{-1}\mathbf{A} = \frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

(b) Consider two bases for $\mathbb{R}^2$, A and B, represented by the columns of matrices $\mathbf{A}$ and $\mathbf{B}$, respectively.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$, that is, find the matrix $\mathbf{T}$, such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A.$$

Solution:

For a given vector $\vec{x}$, we have that

$$\vec{x} = \mathbf{A}\vec{x}_A = \mathbf{B}\vec{x}_B$$

Thus, the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{B}^{-1}\mathbf{A} = \frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$$

(c) Consider two bases for $\mathbb{R}^3$, A and B, represented by the columns of matrices $\mathbf{A}$ and $\mathbf{B}$, respectively.

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Suppose $\vec{x}_A$ represents the coordinates of a vector $\vec{x}$ in basis A and $\vec{x}_B$ represents the coordinates of the same vector in basis B. Write the coordinate transformation that converts $\vec{x}_A$ to $\vec{x}_B$, that is, find the matrix $\mathbf{T}$, such that

$$\vec{x}_B = \mathbf{T}\vec{x}_A$$

Hint: What do you notice about A and B that will simplify this calculation?

Solution:

For a given vector $\vec{x}$, we have that

$$\vec{x} = \mathbf{A}\vec{x}_A = \mathbf{B}\vec{x}_B$$

Thus, the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \mathbf{B}^{-1}\mathbf{A}$$

Since the columns of $\mathbf{B}$ are orthonormal, $\mathbf{B}^{-1} = \mathbf{B}^T$. (This is also true for $\mathbf{A}$.) As a result, we can compute

$$\mathbf{T} = \mathbf{B}^{-1}\mathbf{A} = \mathbf{B}^T\mathbf{A}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}$$

(Actually in this case, $\mathbf{B}$ also happens to be symmetric, so $\mathbf{B} = \mathbf{B}^T = \mathbf{B}^{-1}$, but in general $\mathbf{B} \neq \mathbf{B}^{-1}$ for orthonormal matrices.) Since $\mathbf{A}$ and $\mathbf{B}$ are both orthonormal, $\mathbf{T}$ will be orthonormal as well. (The product of orthonormal matrices is orthonormal.) $\mathbf{T}$ is an example of a coordinate transformation between orthonormal coordinate frames which can always be represented as an orthonormal matrix.

All Rights Reserved. This may not be publicly shared without explicit permission.

4. Completely Normal Eigenvectors

(a) Consider matrix A that has eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$A = \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0.5 & 2.5 & -0.5 \\ 0. & 0. & 4. \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormalize the eigenvectors using Gram-Schmidt to get vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Perform the orthonormalization in the order $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Solution:

$$\vec{u_1}\prime = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$$

$$\vec{u_2}\prime = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_1}\prime, \vec{v_2} \rangle}{\|u_1\prime\|^2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u_3}\prime = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_1}\prime, \vec{v_3} \rangle}{\|u_1\prime\|^2} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_2}\prime, \vec{v_3} \rangle}{\|u_2\prime\|^2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{u}_3$$

(b) Write the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as a linear combination of the eigenvectors. Are any of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ still eigenvectors of the matrix A? Justify your answer.

Solution: $\vec{u}_1$ and $\vec{u}_2$ are both still an eigenvectors, but $\vec{u}_3$ is not. This is because eigenvectors are only the same upto a constant multiplication, not linear combinations with other eigenvectors (in general). Expressing it as a linear combination is exactly what Gram-Schmidt does, so we just need to follow the same process as the previous part but leave terms in symbolic form.

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \vec{v}_1$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \vec{v}_2$$

$$\vec{u}_3 = \vec{v}_3 - \frac{1}{2}\vec{v}_2 - \frac{1}{2}\vec{v}_1$$

(c) Let $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$. Calculate $U^T \cdot U$.

Solution:

$$U^T \cdot U = \begin{bmatrix} \vec{u_1}^T \\ \vec{u_2}^T \\ \vec{u_3}^T \end{bmatrix} \cdot \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

All Rights Reserved. This may not be publicly shared without explicit permission.

Note that the off-diagonal terms $\vec{u_i}^T \vec{u}_j = 0, i \neq j$ because we orthogonalized the vectors. Similarly the diagonal terms are just the norm squared of the vectors, which is 1 from the normalization. Thus

$$U^T \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Prove that if an arbitrary matrix X has orthogonal eigenvectors then X is symmetric i.e. $X^T = X$. You may assume that X exists in $\mathbb{R}^{n\times n}$ and has n linearly independent eigenvectors.

Solution: Since X has the maximum number of eigenvectors, we can write it in terms of it's eigendecomposition. For simplicity assume that the eigenvectors chosen are orthonormal. We are allowed to do this because normalizing a vector is simply scaling it by a constant, and eigenvectors do not change on scaling.

$$X = V\Lambda V^{-1}$$

Since the eigenvectors chosen in V are orthonormal, we know that $V^{-1} = V^T$. Substituting, we get

$$X = V\Lambda V^T$$

$$X^T = (V\Lambda V^T)^T = (V^T)^T \Lambda^T V^T$$

Since Λ is a diagonal matrix, we know that $\Lambda^T = \Lambda$. Thus we have

$$X^T = V\Lambda V^T = X$$