
EECS 16A Designing Information Devices and Systems I

Spring 2019 Homework 4

This homework is due February 22, 2019, at 23:59.

Self-grades are due February 26, 2019, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF “printout” of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

Submit the file to the appropriate assignment on Gradescope.

1. Finding Null Spaces

- (a) Consider the column vectors of any 3×5 matrix. What is the maximum possible number of linearly independent column vectors?

Solution:

Since each column vector is in \mathbb{R}^3 , then there are at most 3 linearly independent column vectors. Hence we can say that the column space of this matrix has at most dimension 3.

For example,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The first three columns span \mathbb{R}^3 , therefore any fourth and fifth columns can be written as a linear combination of the first three columns. This means the columns of \mathbf{A} will always be linearly dependent, no matter what the fourth and fifth columns are.

- (b) Someone performed Gaussian elimination and got the following upper triangular matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of \mathbf{A} . How many unique vectors are required to span the column space of \mathbf{A} ? (This is the dimension of the column space of \mathbf{A})

Solution:

For any vector \vec{x} , $\mathbf{A}\vec{x}$ is a linear combination of the columns of \mathbf{A} . Thus the column space of \mathbf{A} is the space spanned by its columns. We can see that there are only two linearly independent columns

because the third component for each column vector is 0. Therefore a set of linearly independent vectors spanning the range of \mathbf{A} is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$$

This set has two vectors in it, thus the column space has dimension 2.

Another valid set of vectors which span the columns could be,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

Again, this set as two vectors in it, thus the column space has dimension 2.

- (c) Recall that for every vector \vec{x} in the null space of \mathbf{A} , we have $\mathbf{A}\vec{x} = \vec{0}$. The dimension of a the null space is the minimum number of vectors needed to span it. Find vectors that span the null space of \mathbf{A} (the matrix in the previous part). What is the dimension of the null space of \mathbf{A} ? Use the same \mathbf{A} from part b.

Solution:

Finding the null space of \mathbf{A} is the same as solving the following system of linear equations:

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 - 2x_4 + 3x_5 = 0$$

$$x_3 - x_4 + x_5 = 0$$

We have 5 unknowns but only 2 linearly independent equations. Therefore, there are 3 degrees of freedom in the null space. Hence, the dimension of the null space is 3. Note that because of the way Gaussian elimination is performed, x_1 and x_3 only appear once in each equation at the head of their respective rows/equations. Thus, we let x_2 , x_4 and x_5 be free variables a , b and c . Now we rewrite the equations as:

$$x_1 = -a + 2b - 3c$$

$$x_2 = a$$

$$x_3 = b - c$$

$$x_4 = b$$

$$x_5 = c$$

We can then write this in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = a \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of \mathbf{A} is spanned by the vectors:

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) (**Practice**) Now consider the new matrix, \mathbf{B} , which is related to \mathbf{A} ,

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

Find a set of vectors which span the column space of \mathbf{B} . How many unique vectors are required to span the column space of \mathbf{B} ?

Solution:

We see that only the first two column vectors of \mathbf{B} span the column space of \mathbf{B} . The third column does not contribute to the column space of \mathbf{B} . There are only two unique vectors in the set and therefore the column space has dimension 2.

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ 2 \end{bmatrix} \right\}$$

An additional interesting observation is that the dimension of the column space of \mathbf{B} , $C(\mathbf{B})$, and the dimension of the nullspace of \mathbf{A} , $N(\mathbf{A})$, always sum to the number of columns in \mathbf{A} (or the number of rows in \mathbf{B}).

$$\dim(N(\mathbf{A})) + \dim(C(\mathbf{B})) = 5 \quad (1)$$

(e) Find vector(s) that span the null space of the following matrix:

$$\mathbf{C} = \begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix}$$

Solution:

Using Gaussian elimination, we can row-reduce the matrix:

$$\begin{bmatrix} 2 & -4 & 4 & 8 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 1 & -2 & 3 & 6 \\ 2 & -4 & 5 & 10 \\ 3 & -6 & 7 & 14 \end{bmatrix} \xrightarrow{\begin{array}{l} R_1 - R_2 \rightarrow R_2 \\ 2R_1 - R_3 \rightarrow R_3 \\ 3R_1 - R_4 \rightarrow R_4 \end{array}} \begin{bmatrix} 1 & -2 & 2 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\begin{array}{l}
 2R_2 + R_1 \rightarrow R_1 \\
 R_2 - R_3 \rightarrow R_3 \\
 R_2 - R_4 \rightarrow R_4 \\
 \underbrace{\hspace{1.5cm}}_{\Rightarrow}
 \end{array}
 \begin{bmatrix}
 1 & -2 & 0 & 0 \\
 0 & 0 & -1 & -2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{bmatrix}
 \xrightarrow{-R_2 \rightarrow R_2}
 \begin{bmatrix}
 1 & -2 & 0 & 0 \\
 0 & 0 & 1 & 2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{bmatrix}$$

Vectors in the null space satisfy the following equations:

$$\begin{bmatrix}
 1 & -2 & 0 & 0 \\
 0 & 0 & 1 & 2 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0
 \end{bmatrix}
 \begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4
 \end{bmatrix}
 = 0$$

$$\begin{aligned}
 x_1 - 2x_2 &= 0 \\
 x_3 + 2x_4 &= 0
 \end{aligned}$$

We then set x_2 and x_4 to be free variables and substitute in:

$$\begin{aligned}
 x_1 &= 2a \\
 x_2 &= a \\
 x_3 &= -2b \\
 x_4 &= b
 \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix}
 x_1 \\
 x_2 \\
 x_3 \\
 x_4
 \end{bmatrix}
 = a
 \begin{bmatrix}
 2 \\
 1 \\
 0 \\
 0
 \end{bmatrix}
 + b
 \begin{bmatrix}
 0 \\
 0 \\
 -2 \\
 1
 \end{bmatrix}$$

Therefore, the null space of the matrix is spanned by the vectors:

$$\left\{
 \begin{bmatrix}
 2 \\
 1 \\
 0 \\
 0
 \end{bmatrix},
 \begin{bmatrix}
 0 \\
 0 \\
 -2 \\
 1
 \end{bmatrix}
 \right\}$$

Notice the same pattern as before in terms of the relationship of the null space basis found to the fully row-reduced matrix.

2. Mechanical Eigenvalues and Eigenvectors

Find the eigenvalues and their eigenspaces — give a basis for the eigenspace when it is more than 1 dimensional

$$\text{(a)} \quad \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Solution:

There are two ways to do this.

First, we can do it by inspection. We can see that this matrix multiplies everything in the first coordinate by 5 and everything in the second by 2. Consequently, when given $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, it will return 2 times the input.

And when given $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, it will return 5 times the input vector.

Alternatively, we can use determinants.

$$\begin{vmatrix} 5-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(2-\lambda) - 0 = 0$$

This is already factored for you! We see that, by definition, diagonal matrices have their eigenvalues on the diagonal.

$\lambda = 5$:

$$\mathbf{A}\vec{x} = 5\vec{x} \implies (\mathbf{A} - 5\mathbf{I}_2)\vec{x} = \vec{0}$$

$$\left(\begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

where x is a free variable.

The eigenspace for $\lambda = 5$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$.

$\lambda = 2$:

$$\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

where y is a free variable.

The eigenspace for $\lambda = 2$ is $\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

(b) $\begin{bmatrix} 22 & 6 \\ 6 & 13 \end{bmatrix}$

Solution:

Here, it is hard to guess the answers.

$$(22-\lambda)(13-\lambda) - 36 = 0 \implies \lambda = 10, 25$$

$\lambda = 10$:

$$\begin{bmatrix} 12 & 6 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x + y = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$

where x is a free variable.

The eigenspace for $\lambda = 10$ is $\text{span}\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$.

$\lambda = 25$:

$$\begin{bmatrix} -3 & 6 \\ 6 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2y = x \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$$

where y is a free variable.

The eigenspace for $\lambda = 25$ is $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

(c) **(PRACTICE)** $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Solution:

This can also be seen by inspection. The matrix is clearly not invertible since the first two rows are linearly dependent. Therefore, there must be a 0 eigenvalue. This has the eigenvector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

The other eigenvector can be seen by noticing that the second row is twice the first. So $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a good guess to try and indeed it works with $\lambda = 5$.

Alternatively, we can brute force it.

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$\begin{aligned} (1-\lambda)(4-\lambda) - 4 &= 0 \\ \lambda^2 - 5\lambda &= 0 \implies \lambda(\lambda - 5) = 0 \\ \lambda &= 0, 5 \end{aligned}$$

$\lambda = 0$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$x + 2y = 0 \implies y = -\frac{1}{2}x \implies \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The eigenspace for $\lambda = 0$ is $\text{span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$.

$\lambda = 5$:

$$\begin{bmatrix} 1-5 & 2 \\ 2 & 4-5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

$$2x - y = 0 \implies y = 2x \implies \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenspace for $\lambda = 5$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

(d) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (What special matrix is this?)

Solution:

This is a rotation matrix (counterclockwise by 30 degrees).

$$\begin{vmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{vmatrix} = 0$$

$$\left(\frac{\sqrt{3}}{2} - \lambda\right)^2 + \frac{1}{4} = 0 \implies \lambda = \frac{\sqrt{3} \pm i}{2}$$

$$\lambda = \frac{\sqrt{3}+i}{2}:$$

$$\begin{bmatrix} \frac{-i}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{-i}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x - iy = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix}$$

where y is a free variable. The eigenspace for $\lambda = \frac{\sqrt{3}+i}{2}$ is $\text{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$.

$$\lambda = \frac{\sqrt{3}-i}{2}:$$

$$\begin{bmatrix} \frac{i}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + iy = 0 \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ ix \end{bmatrix}$$

where x is a free variable. The eigenspace for $\lambda = \frac{\sqrt{3}-i}{2}$ is $\text{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$.

(e) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

Solution:

$$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 = 0 \implies \lambda = 2, 2$$

$$\lambda = 2:$$

$$\begin{bmatrix} 2-2 & 0 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

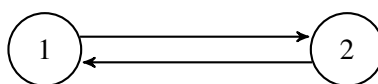
Now, both x, y are free variables. This means that our eigenspace for $\lambda = 2$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$.

3. Counting The Paths of a Random Surfer

In homework and discussion, we have discussed the behavior of water flowing in reservoirs and the people flowing in social networks. We now consider the behavior of a random web-surfer who jumps from webpage to webpage. We would like to know how many possible paths there are for a random surfer to get from one webpage to another webpage. To do this, we represent the webpages as a graph.

If webpage 1 has a link to webpage 2, we have a directed edge from webpage 1 to webpage 2. This graph can further be represented by what is known as an “adjacency matrix”, \mathbf{A} , with elements a_{ij} . We define $a_{ij} = 1$ if there is link from page i to page j . Note the ordering of the indices! Matrix operations on the adjacency matrix make it very easy to compute the number of paths to get from a particular webpage i to webpage j .

Consider the following graphs.



Graph A

- (a) Based on this definition, the “adjacency matrix” for graph A, will be,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (2)$$

The element a_{ji} of \mathbf{A} gives the number of one-hop paths from website i to website j . Similarly, the elements of \mathbf{A}^2 give the number of two-hop paths from website i to website j . How many one-hop paths are there from webpage 1 to webpage 2? How many two-hop paths are there from webpage 1 to webpage 2? How about three-hop paths?

Solution:

We take the n^{th} power of the adjacency matrix to determine how many n -hops paths exist between the pages.

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Therefore, there is 1 one-hop path between webpage 1 and webpage 2 (which can be checked trivially).

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

There are no two-hop paths between webpage 1 and webpage 2! This matches the structure of the graph since two hops will always get the websurfer back to the page they started from.

Why does this work? Let’s look at what the \mathbf{A}^2 matrix multiplication does:

The first element (describing the path to get to and from node 1 in two hops) is (number of paths from node 1 to node 1)²+(number of paths from node 2 to node 1)(number of paths from node 1 to node 2). This is $0 + (1)(1) = 1$. The result is therefore the sum of any self-loops and the number of paths going to node 2 and back. A similar formula applies for the n^{th} powers.

This is because the concatenation of two paths is a valid path if one ends where the other begins. So the number of n -hop paths from i to j must in fact be the sum, over all intermediate pages k , of the number of ℓ -hop paths from i to k times the number of $(n - \ell)$ -hop paths from k to j . This is precisely what matrix multiplication does.

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

There is 1 three-hop path between webpage 1 and webpage 2. Note that $\mathbf{A}^3 = \mathbf{A}$.

- (b) This path counting aspect is very related to the steady-state frequency for the fraction of people for each webpage. The steady-state frequency for a graph of websites is related to the eigenspace associated with eigenvalue 1 for the “transition matrix” of the graph.

The “transition matrix”, \mathbf{T} , is slightly different from the “adjacency matrix”. Its values, t_{ji} , are the *proportion* of the people who are at website i that click the link for website j . We assume people divide equally among the links on the website (e.g. if there are three links on a website, $\frac{1}{3}$ of the people will click each link).

Once computed, an eigenvector with eigenvalue 1 will have values which correspond to the steady-state frequency for the fraction of people for each webpage. When this eigenvector's values are made to sum to one (to conserve people), the i^{th} element of the eigenvector corresponds to the fraction of people on the i^{th} website.

For graph A, what are the steady-state frequencies for the two webpages?

Solution:

To determine the steady-state frequencies for the two pages, we need to find the appropriate eigenvector of the transition matrix. In this case, we are trying to determine the proportion of people who would be on a given page at steady state.

The transition matrix of graph A:

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3)$$

To determine the eigenvalues of this matrix:

$$\det \left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \right) = \lambda^2 - 1 = 0 \quad (4)$$

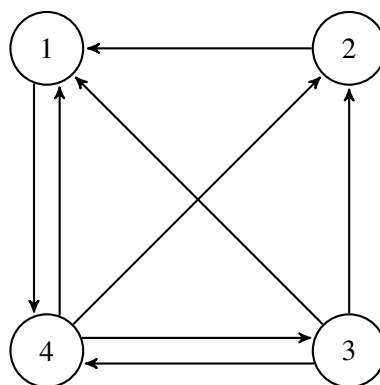
$\lambda = 1, -1$. The steady state vector is the eigenvector that corresponds to $\lambda = 1$. To find the eigenvector,

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (5)$$

The sum of the values of the vector should equal 1, so our conditions are:

$$\begin{aligned} v_1 + v_2 &= 1 \\ v_1 &= v_2 \end{aligned}$$

The steady-state frequency eigenvector is $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ and each webpage has a steady-state frequency of 0.5.



Graph B

(c) Write out the adjacency matrix for graph B.

Solution:

Let \mathbf{B} be the adjacency matrix for graph B. Then,

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- (d) For graph B, how many two-hop paths are there from webpage 1 to webpage 3? How many three-hop paths are there from webpage 1 to webpage 2? You may use IPython for this.

Solution:

Using the same procedure as part (b),

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

There is 1 two-hop path from webpage 1 to webpage 3. Let's look at what the \mathbf{B}^2 matrix multiplication does:

The B_{31} element (describing the path to get from webpage 1 to webpage 3) is

(paths from node 1 to node 3)(paths from node 1 to node 1)+(paths from node 2 to node 3)(paths from node 1 to node 2)+(paths from node 3 to node 3)(paths from node 1 to node 3)+(paths from node 4 to node 3)(paths from node 1 to node 4)+(paths from node 5 to node 3)(paths from node 1 to node 5).

This is $0 + 0 + 0 + 1 = 1$. A similar formula applies for the n^{th} powers.

For three hop paths,

$$\mathbf{B}^3 = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 2 \end{bmatrix}$$

There is 1 three-hop path from webpage 1 to webpage 2.

- (e) For graph B, what are the steady-state frequencies for the webpages? You may use IPython and the Numpy command `numpy.linalg.eig` for this.

Solution:

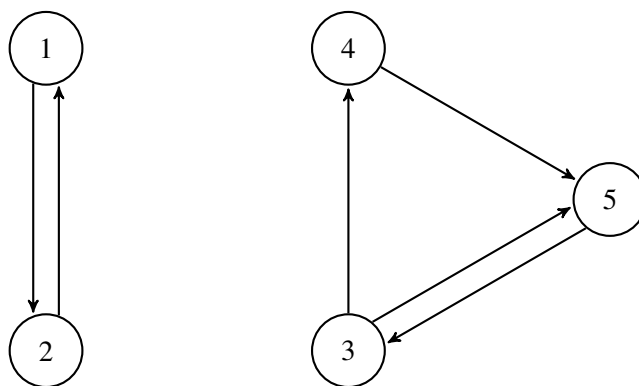
To determine the steady-state frequencies, we need to create the transition matrix \mathbf{T} first.

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} & 0 \end{bmatrix}$$

The eigenvector associated with eigenvalue 1 is $[-0.61 \quad -0.31 \quad -0.23 \quad -0.69]^T$ (found using IPython).

Scaling it appropriately so the elements add to 1, we get $[\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{8} \quad \frac{3}{8}]^T$

These are the steady-state frequencies for the pages.



Graph C

- (f) Write out the adjacency matrix for graph C.

Solution:Let \mathbf{C} be the adjacency matrix for graph C. Then,

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- (g) For graph C, how many paths are there from webpage 1 to webpage 3?

Solution:There are no paths from webpage 1 to webpage 3 (and no n -hop paths either).

- (h) (
- PRACTICE**
-) Find the eigenspace that corresponds to the steady-state for graph C. How many independent systems (disjoint sets of webpages) are there in graph C versus in graph B? What is the dimension of the eigenspace corresponding to the steady-state for graph C?

Solution:

The transition matrix for graph C is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$$

The eigenvalues of this graph are $\lambda = 1, 1, -1, -\frac{1}{2} + -\frac{i}{2}, -\frac{1}{2} - -\frac{i}{2}$ (found using IPython). The eigenspace associated with $\lambda = 1$ is span by the vectors $[0 \ 0 \ 0.4 \ 0.2 \ 0.4]^T$ and $[0.5 \ 0.5 \ 0 \ 0 \ 0]^T$. While, any linear combination of these vectors is an eigenvector, these two particular vectors have a nice interpretation.

The first eigenvector describes the steady-state frequencies for the last three webpages, and the second vector describes the steady-state frequencies for the first two webpages. This makes sense since there are essentially “two internets”, or two disjoint set of webpages. Surfers cannot transition between the two, so you cannot assign steady-state frequencies to webpage 1 and webpage 2 relative to the rest. This is why the eigenspace corresponding to the steady-state has dimension 2.

Assuming that each set of steady-state frequencies needs to add to 1, the first assigns steady-state frequencies of 0.4, 0.2, 0.4 to webpage 3, webpage 4, and webpage 5, respectively. The second assigns steady-state frequencies of 0.5 to both webpage 1 and webpage 2.

4. Traffic Flows

Your goal is to measure the flow rates of vehicles along roads in a town. However, it is prohibitively (too) expensive to place a traffic sensor along every road. You realize, however, that the number of cars flowing into an intersection must equal the number of cars flowing out. You can use this “flow conservation” to determine the traffic along all roads in a network by only measuring flow along only some roads. In this problem, we will explore this concept.

- (a) Let's begin with a network with three intersections, A , B and C . Define the flow t_1 as the rate of cars (cars/hour) on the road between B and A , flow t_2 as the rate on the road between C and B , and flow t_3 as the rate on the road between C and A .

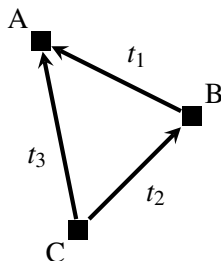


Figure 1: A simple road network.

(Note: The directions of the arrows in the figure are the way that we define positive flow by convention. For example, if there were 100 cars per hour traveling from A to C , then $t_3 = -100$.)

We assume the “flow conservation” constraints: the net number of cars per hour flowing into each intersection is zero. For example at intersection B , we have the constraint $t_2 - t_1 = 0$. The full set of constraints (one per intersection) is:

$$\begin{cases} t_1 + t_3 = 0 \\ t_2 - t_1 = 0 \\ -t_3 - t_2 = 0 \end{cases}$$

As mentioned earlier, we can place sensors on a road to measure the flow through it, but we have a limited budget, and we would like to determine all of the flows with the smallest possible number of sensors.

Suppose for the network above we have one sensor reading, $t_1 = 10$. Can we figure out the flows along the other roads? (That is, the values of t_2 and t_3). If we can, find the values of t_2 and t_3 .

Solution:

Yes, since we know that $t_1 = t_2 = -t_3$, so we must have $t_2 = 10$ and $t_3 = -10$.

- (b) Now suppose we have a larger network, as shown in Figure 2.

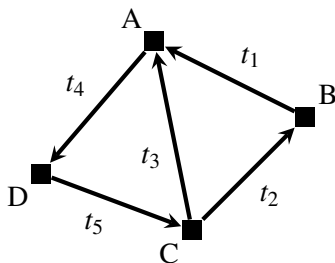


Figure 2: A larger road network.

We would again like to determine the traffic flows on all roads, using measurements from some sensors. A Berkeley student claims that we need two sensors placed on the roads AD (measuring t_4) and BA (measuring t_1). A Stanford student claims that we need two sensors placed on the roads CB (measuring t_2) and BA (measuring t_1). Write out the system of linear equations that represents this flow graph. Is

it possible to determine all traffic flows, $[t_1, t_2, t_3, t_4, t_5]^T$, with the Berkeley student's suggestion? How about the Stanford student's suggestion?

Solution: Since we have 4 intersections, we can write 4 linear equations describing the flows into and out of each intersection. We know that the flows into and out of an intersection must sum to 0. The set of linear equations that represents this flow graph is:

$$\begin{cases} t_1 + t_3 - t_4 = 0 \\ t_2 - t_1 = 0 \\ t_5 - t_2 - t_3 = 0 \\ t_4 - t_5 = 0 \end{cases}$$

The Stanford student is wrong (obviously). Observing t_1 and t_2 is not sufficient, as t_3 , t_4 and t_5 can still not be uniquely determined. Specifically, for any $\alpha \in \mathbb{R}$, the following flow satisfies the constraints and the measurements:

$$\begin{aligned} t_4 &= \alpha \\ t_5 &= \alpha \\ t_3 &= \alpha - t_1 \end{aligned}$$

On the other hand, if we're given t_1 and t_4 , we can uniquely solve for all the traffic densities as follows since we know the flow conservation constraints. We know that t_2 is the same as t_1 and that t_4 is the same as t_5 since the flow going into B and D must equal the flow going out. The flow into A , $t_1 + t_3$, must equal the flow going out, t_4 , so:

$$\begin{aligned} t_2 &= t_1 \\ t_5 &= t_4 \\ t_3 &= t_4 - t_1 \end{aligned}$$

This is related to the fact that t_1 and t_4 are parts of different cycles in the graph, whereas t_1 and t_2 are in the same cycle, so measuring both of them would not give additional information.

- (c) We would like a more general way of determining the possible traffic flows in a network. Suppose we

write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. As a first step, let us try to write all the flow

conservation constraints (one per intersection) as a matrix equation.

Construct a 4×5 matrix \mathbf{B} such that the equation $\mathbf{B}\vec{t} = \vec{0}$

$$\begin{bmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix} \mathbf{B} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

represents the flow conservation constraints for the network in Figure 2.

Hint: Each row is the constraint of an intersection. You can construct \mathbf{B} using only 0, 1, and -1 entries. This matrix is called the **incidence matrix**. What constraint does each column of \mathbf{B} represent?

Solution:

$$\mathbf{B} = \begin{array}{ccccc} \left[\begin{array}{ccccc} 1 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right] & \begin{array}{l} A \\ B \\ C \\ D \end{array} \\ \begin{array}{ccccc} t_1 & t_2 & t_3 & t_4 & t_5 \end{array} & \end{array}$$

(The rows of this matrix can be in any order and your solution can differ by a factor of -1.) Each row represents an intersection, and each column represents a road between two intersections. Each 1 on a row represents a road flowing into an intersection, and each -1 represents a road flowing out of an intersection. Each -1 in a column represents the source intersection of a road (where the arrow starts), and each 1 in a column represents the destination intersection of a road (where the arrow ends).

Each column of \mathbf{B} must sum to 0. The columns represent the flows into and out of an intersection. Since we know that each intersection's flows in equal flows out, we know that each column must sum to 0

- (d) Again, suppose we write the traffic flow on all roads as a vector $\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{bmatrix}$. Then, determine the subspace

of all valid traffic flows for the network of Figure 2. Specifically, express this space as the span of two linearly independent vectors.

Hint: Use the claim of the Berkeley student in part (b). Justify why you can use their claim. Then write all valid flows as a vector in terms of $t_1 = \alpha$ and $t_4 = \beta$.

Solution: To determine the subspace of traffic flows for the above network, use the solution in the previous part to see what \vec{t} looks like in terms of $t_1 = \alpha$ and $t_4 = \beta$:

$$\vec{t} = \begin{bmatrix} \alpha \\ \alpha \\ \beta - \alpha \\ \beta \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \vec{u}_1 + \beta \vec{u}_2$$

Clearly, \vec{u}_1 and \vec{u}_2 are linearly independent, and the space of all possible traffic flows is the span of them.

Note to grader: If you used Gaussian Elimination in this step and found another basis for the nullspace, that is also fine for this part. The student did not need to use the hint.

Note: We show here, for your reference, that the space of all possible traffic flows is a subspace. You don't need to include this proof in your solution. Suppose we have a set of valid flows \vec{t} . Then, for any intersection, the total flow into it is the same as the total flow out of it. If we scale \vec{t} by a constant a , each t_i will also get scaled by a . The total flows into and out of the intersection would be scaled by the same amount and remain equal to each other. Thus any scaling of a valid flow is still a valid flow. Suppose now we add valid flows \vec{t}_1 and \vec{t}_2 to get $\vec{t} = \vec{t}_1 + \vec{t}_2$. For any intersection I ,

$$\begin{aligned} \text{total flow into } I &= \text{total flow into } I \text{ from } \vec{t}_1 + \text{total flow into } I \text{ from } \vec{t}_2 \\ \text{total flow out of } I &= \text{total flow out of } I \text{ from } \vec{t}_1 + \text{total flow out of } I \text{ from } \vec{t}_2 \end{aligned}$$

Since the total flow into I from \vec{t}_1 is the same as the total flow out of I from \vec{t}_1 and similarly for \vec{t}_2 , the total flow into I is the same as the total flow out of I . Therefore, the sum of any two valid flows is still a valid flow. Also, $\vec{t} = \vec{0}$ is a valid flow. Therefore the set of valid flows forms a subspace.

- (e) Notice that the set of all vectors \vec{t} that satisfy $\mathbf{B}\vec{t} = \vec{0}$ is exactly the null space of the matrix \mathbf{B} . That is, we can find all valid traffic flows by computing the null space of \mathbf{B} . Use Gaussian elimination to determine the dimension of the null space of \mathbf{B} and compute a basis for the null space. Does this match your answer to part (d)?

Challenge (optional): Can you interpret the dimension of the null space of \mathbf{B} for the road networks of Figure 1 and Figure 2?

Solution:

After row-reducing, we get the following matrix:

$$\begin{bmatrix} +1 & 0 & +1 & 0 & -1 \\ 0 & +1 & +1 & 0 & -1 \\ 0 & 0 & 0 & +1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have 5 unknowns but from the matrix above, we can see that we have only 3 non-zero rows. Therefore we have 2 free parameters. So, the dimension of the null space is 2. We can find the following basis for the null space by designating $t_3 = a$ and $t_5 = b$ ($a, b \in \mathbb{R}$) as the free variables and solving for the parametric solution:

$$a \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

This does not match the answer in the earlier part because these are two different bases, but the null space they span is the same.

Note to grader: It is fine to give yourself full credit as long as you found a valid basis for the null space using Gaussian Elimination. It doesn't have to be this particular one. Also, if you have the same answer for part (d) and part (e), that is fine (assuming the answer is a valid basis for the null space).

Challenge Solution: By itself, the first vector weighted by a is a vector corresponding to the small cycle in the graph. The second vector weighted by b corresponds to the bigger cycle. These two cycles are still independent of each other, which is why the dimension of the null space can be interpreted as the number of "independent cycles" in the graph.

- (f) Now let us analyze more general road networks. Say there is a road network graph G , with incidence matrix \mathbf{B}_G . If \mathbf{B}_G has a k -dimensional null space, does this mean measuring the flows along **any** k roads is always sufficient to recover the exact flows? Prove or give a counterexample.

Hint: Consider the Stanford student from part (b).

Solution:

No, consider the network of Figure 2. The corresponding incidence matrix has a $k = 2$ dimensional null space, as you showed in part (e). However, measuring t_1 and t_2 (as the Stanford student suggested) is not sufficient, as you showed in part (b).

- (g) Let G be a network of n roads with the incidence matrix \mathbf{B}_G , which has a k -dimensional null space. We would like to characterize exactly when it is sufficient to measure a set of k roads to recover the exact

flow along all roads.

To do this, it will help to generalize the problem and consider measuring *linear combinations* of flows. Let t_i be the flow on one road. We measure some linear combination of t_i 's or $m_0 \cdot t_0 + m_1 \cdot t_1 + \dots + m_n \cdot t_n$. Now we measure many of these linear combinations, which we will represent using matrix vector multiplication. Then, making k measurements is equivalent to observing the vector $\mathbf{M}\vec{t}$ for some $k \times n$ “measurement matrix” \mathbf{M} .

For example, for the network of Figure 2, the measurement matrix corresponding to measuring t_1 and t_4 (as the Berkeley student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, the measurement matrix corresponding to measuring t_1 and t_2 (as the Stanford student suggests) is:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

For general networks G and measurements \mathbf{M} , give a condition for when the exact traffic flows can be recovered in terms of the null space of \mathbf{M} and the null space of \mathbf{B}_G .

Hint: Recovery will fail iff (if and only if) there are two valid flows with the same measurements, that is, there exist distinct \vec{t}_1 and \vec{t}_2 satisfying the flow conservation constraints, such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$. Can you express this in terms of the null spaces of \mathbf{M} and \mathbf{B}_G ?

Solution:

As stated in the hint, we cannot uniquely determine the flow iff (if and only if) there are two valid flows that yield the same set of measurements. That is, there should not be two *distinct* valid flows \vec{t}_1 and \vec{t}_2 , such that $\mathbf{M}\vec{t}_1 = \mathbf{M}\vec{t}_2$, or equivalently, such that $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = \vec{0}$.

The set of valid flows is the null space of \mathbf{B}_G , denoted $\text{Null}(\mathbf{B}_G)$. So recovery fails if $\mathbf{M}(\vec{t}_1 - \vec{t}_2) = \vec{0}$ for some $\vec{t}_1, \vec{t}_2 \in \text{Null}(\mathbf{B}_G)$, with $\vec{t}_1 \neq \vec{t}_2$. The set of valid flows is a subspace, so we can equivalently state this as: Recovery fails iff $\mathbf{M}\vec{t} = \vec{0}$ for some $\vec{t} \neq \vec{0}$, $\vec{t} \in \text{Null}(\mathbf{B}_G)$.

In other words, *there should be no vector $\vec{t} \neq \vec{0}$ that is both in the null space of \mathbf{B}_G and in the null space of \mathbf{M} .*

This can also be stated as: *We can recover the exact traffic flows iff (if and only if) the null space of \mathbf{B}_G does not non-trivially intersect the null space of \mathbf{M} .*

Full credit for stating any condition that is equivalent to this, using the null spaces of \mathbf{M} and \mathbf{B}_G .

- (h) **Challenge (optional):** If the incidence matrix \mathbf{B}_G has a k -dimensional null space, does this mean we can **always pick a set of k roads** such that measuring the flows along these roads is sufficient to recover the exact flows? Prove or give a counterexample.

Solution:

Yes.

Let \mathbf{U} be a matrix whose columns form a basis of the null space of \mathbf{B}_G , as above. The k columns of \mathbf{U} are linearly independent since they form a basis. Since there are k linearly independent columns, when we run Gaussian elimination on \mathbf{U} , we must get k pivots. (Recall that “pivot” is the technical term for being able to row-reduce and turn a column into something that has exactly one 1 in it. The pivot is the entry that we found and turned into that 1.)

Therefore, the row space of \mathbf{U} is k dimensional since there are some k linearly independent rows in \mathbf{U} — namely the ones where we found pivots. Choose to measure the roads corresponding to these rows.

This will work because:

For a given valid flow $\vec{f} = \mathbf{U}\vec{x}$, the results of measuring this flow vector are $\mathbf{U}^{(k)}\vec{x}$, where the matrix $\mathbf{U}^{(k)}$ is some k linearly independent rows of \mathbf{U} . By construction, the $k \times k$ matrix $\mathbf{U}^{(k)}$ has all linearly independent rows, so we can invert $\mathbf{U}^{(k)}$ to find \vec{x} from $\mathbf{U}^{(k)}\vec{x}$ and then recover the flows along all the edges as $\mathbf{U}\vec{x}$.

This isn't the only set of k roads that will work. But it does provide a set of k roads that are guaranteed to work.

5. (PRACTICE) Codes Revisited

Alice and Bob are back and they've successfully figured out how to avoid dropping symbols when sending messages. In this problem, Alice is using a similar encoding scheme as last time where she uses vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ to encode her message $[a \ b \ c]^T$. (Assume Bob knows the vectors, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, that Alice is using.) Namely, she tries to send \vec{k} :

$$\vec{k} = \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ - & \vec{v}_3^T & - \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (6)$$

Unfortunately, their arch-nemesis, Eve, is trying to interfere with Alice's messages to Bob and has found a way to add noise to the transmission! But, Eve's interference must pass through a linear transformation, \mathbf{U} , before the interference hits Alice's message. Now instead of seeing \vec{k} , Bob is receiving \vec{y} :

$$\vec{y} = \begin{bmatrix} - & \vec{v}_1^T & - \\ - & \vec{v}_2^T & - \\ - & \vec{v}_3^T & - \end{bmatrix} \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \right) \quad (7)$$

where Eve inserts $[p \ q \ r]^T$ and it undergoes a transformation by the matrix \mathbf{U} and then undergoes the same transformation as Alice's original 3-symbol vector. There are two ways Eve's meddling can mess up the transmission:

- If Bob receives $\vec{0}$, he doesn't even realize he's getting a message
- Bob receives a nonzero transmission but can't determine the original $[a \ b \ c]^T$

(a) (PRACTICE) Alice is using the following vectors for her encoding scheme

$$\vec{v}_1^T = [1 \ 2 \ 0], \vec{v}_2^T = [0 \ 0 \ 1], \vec{v}_3 = [1 \ 2 \ 1]$$

If Eve is not interfering ($p = q = r = 0$), will Bob be able to uniquely determine what a , b , and c are?

Solution:

Nope! We can either use Gaussian elimination to see that there's no unique solution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} \leftarrow R_3 - R_2 \mapsto R_3 \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_1 - R_3 \mapsto R_3 \quad (8)$$

or we can see that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$. Therefore, the rows are linearly dependent and we are not getting 3 unique measurements. Since we have 3 unknowns, we need at least 3 measurements to find a unique solutions. Alice chose a bit of an unfortunate encoding scheme; even if Eve isn't interfering, Bob won't be able to determine what she's sending!

- (b) **(PRACTICE)** Eve decides to change her strategy—now she's sending her interference through its own transformation before adding it to Alice's message. Bob receives \vec{y} according to Equation (7).

Eve's interference is transformed by the vectors:

$$\vec{u}_1^T = [1 \ 2 \ 0], \vec{u}_2^T = [0 \ 0 \ 1], \vec{u}_3 = [1 \ 2 \ 1] \quad (9)$$

Find the null space of $\mathbf{U} = \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix}$

Solution:

Calculating the null space of \mathbf{U} :

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow R_3 - R_2 - R_1 \mapsto R_3$$

$$p + 2q = 0$$

$$r = 0$$

$$\text{null}(\mathbf{U}) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

- (c) **(PRACTICE)** Given \mathbf{U} from Equation (9), find p , q , and r such that Bob is guaranteed to never realize he's receiving a message, regardless of Alice's encoding scheme, i.e. $\vec{y} = \vec{0}$ (this corresponds to the fact that the null space of any matrix always includes the zero vector).

State any constraints on a , b , and c which are necessary for Eve's cancellation to work. In other words, find $[p \ q \ r]^T$ and define any restrictions on a , b , and c such that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \vec{0}$$

Do not take into account the vectors Alice is using for her encoding.

Solution:

Rearranging the equation a bit,

$$\begin{bmatrix} - & \vec{u}_1^T & - \\ - & \vec{u}_2^T & - \\ - & \vec{u}_3^T & - \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} -a \\ -b \\ -c \end{bmatrix}$$

we can use Gaussian elimination to determine what p , q , and r are with respect to a , b , and c .

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -a \\ 0 & 0 & 1 & -b \\ 1 & 2 & 1 & -c \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -a \\ 0 & 0 & 1 & -b \\ 0 & 0 & 0 & a+b-c \end{array} \right]$$

In other words,

$$\begin{aligned} p + 2q &= -a \\ r &= -b \\ 0 &= a + b - c \end{aligned}$$

For this to be a consistent set of equations and for Eve to even have a chance of canceling out Alice's original message, $a + b - c = 0$. To have Eve's interference cancel out Alice's symbols, $p + 2q = -a$ and $r = -b$.

- (d) **(PRACTICE)** Realizing her scheme from (a) was flawed, Alice has also chosen to change things up and is using new vectors for her encoding:

$$\vec{v}_1^T = [1 \ 2 \ 0], \vec{v}_2^T = [0 \ 0 \ 1], \vec{v}_3 = [1 \ 3 \ 1]$$

For each of the following cases, determine whether Bob will receive a message at all and—if he does—whether he'll be able to correctly and uniquely determine what a , b , and c are. Justify your answer.

Do not use IPython to solve this problem. *Hint: Use your answers from (b) and (c)*

- i. $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$
- ii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
- iii. $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

Solution:

To save on some writing, we'll define the following:

$$\vec{x}_{\text{Alice}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \vec{x}_{\text{Eve}} = \begin{bmatrix} p \\ q \\ r \end{bmatrix},$$

From Equation (7), we can use the distributive property of matrix multiplication

$$\mathbf{V}(\vec{x}_{\text{Alice}} + \mathbf{U}\vec{x}_{\text{Eve}}) = \mathbf{V}\vec{x}_{\text{Alice}} + \mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}}$$

For Bob to receive Alice's uncorrupted transmission, he should only receive $\mathbf{V}\vec{x}_{\text{Alice}}$. In other words, if $\mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$, Bob will receive exactly what Alice intended. Because the columns of \mathbf{V} are linearly independent, we know from lecture that means that $\mathbf{V}\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$ only when $\mathbf{U}\vec{x}_{\text{Eve}} = \vec{0}$. Written another way, Eve's interference will have no effect if \vec{x}_{Eve} falls in the null space of \mathbf{U} calculated in part (b).

For Bob to not realize that he's receiving a message, i.e. for him to receive $\vec{0}$, $\vec{x}_{\text{Alice}} + \mathbf{U}\vec{x}_{\text{Eve}}$ must fall in the null space of \mathbf{V} . Again, the null space of \mathbf{V} is just the zero vector, so we can use our answer from part (c) to determine if Eve's interference is canceling out Alice's message.

- i. Bob will receive a message and will be able to determine what a , b , and c are. There are two parts to this: (1) Eve's interference vector falls in $\text{null}(\mathbf{U})$, she's adding zero interference to Alice's original transmission.

$$\begin{aligned}\mathbf{V}(\vec{v}_{\text{Alice}} + \mathbf{U}\vec{v}_{\text{Eve}}) &= \mathbf{V}(\vec{v}_{\text{Alice}} + \vec{0}) \\ &= \mathbf{V}\vec{v}_{\text{Alice}}\end{aligned}$$

(2) The columns of \mathbf{V} are linearly independent with 3 equations and 3 unknowns, meaning Bob will be able to uniquely find the elements of \vec{x}_{Alice} .

- ii. Bob will receive a message and be unable to determine what a , b , and c are. We know that \vec{v}_{Eve} does not fall in the null space of \mathbf{U} , so now we should check if Bob will receive a message by using our answer from part (c). First, we see that $a + b - c = 1 + 1 - 2 = 0$, so it's not impossible for Eve to cancel Alice's message. $p + 2q = 1 + 3 = 4 \neq -a$, so Bob receives a message, but it's been garbled by Eve's interference.
- iii. Bob won't realize he's being sent a message. Using our answer from part (c), we can see that Bob will receive $\vec{0}$, meaning he won't even be aware that Alice is attempting to transmit.

6. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.