EECS 16A Designing Information Devices and Systems I
Spring 2019 Homework 5

This homework is due March 1, 2019, at 23:59.
Self-grades are due March 5, 2019, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

- `hw5.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit the file to the appropriate assignment on Gradescope.

1. The Dynamics of Romeo and Juliet's Love Affair

In this problem, we will study a discrete-time model of the dynamics of Romeo and Juliet's love affair—adapted from Steven H. Strogatz's original paper, *Love Affairs and Differential Equations*, Mathematics Magazine, 61(1), p.35, 1988, which describes a continuous-time model.

Let $R[n]$ denote Romeo's feelings about Juliet on day n, and let $J[n]$ denote Juliet's feelings about Romeo on day n. The sign of $R[n]$ (or $J[n]$) indicates like or dislike. For example, if $R[n] > 0$, it means Romeo likes Juliet. On the other hand, $R[n] < 0$ indicates that Romeo dislikes Juliet. $R[n] = 0$ indicates that Romeo has a neutral stance towards Juliet.

The magnitude (i.e. absolute value) of $R[n]$ (or $J[n]$) represents the intensity of that feeling. For example, a larger $|R[n]|$ means that Romeo has a stronger emotion towards Juliet (love if $R[n] > 0$ or hatred if $R[n] < 0$). Similar interpretations hold for $J[n]$.

We model the dynamics of Romeo and Juliet's relationship using the following linear system:

$$R[n+1] = aR[n] + bJ[n], \quad n = 0, 1, 2, \ldots$$

and

$$J[n+1] = cR[n] + dJ[n], \quad n = 0, 1, 2, \ldots,$$

which we can rewrite as

$$\vec{s}[n+1] = \mathbf{A}\vec{s}[n],$$

where $\vec{s}[n] = \begin{bmatrix} R[n] \\ J[n] \end{bmatrix}$ denotes the state vector and $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the state transition matrix for our dynamic system model.

The selection of the parameters a, b, c, d results in different dynamic scenarios. The fate of Romeo and Juliet's relationship depends on these model parameters (i.e. a, b, c, d) in the state transition matrix and the initial state ($\vec{s}[0]$). In this problem, we'll explore some of these possibilities.

(a) Consider the case where $a+b=c+d$ in the state-transition matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Show that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_1. Show that

$$\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$, and determine its corresponding eigenvalue λ_2. Now, express the first and second eigenvalues and their eigenspaces in terms of the parameters $a, b, c,$ and d.

Hint: You could use the characteristic polynomial approach to find the eigenvalues and eigenvectors. You may find it easier to use the following approach instead:

- First find λ_1 by showing $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.
- Then find λ_2 by showing $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$ is an eigenvector of $\mathbf{A}$.

Solution:

$$\begin{aligned} \mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} a+b \\ c+d \end{bmatrix} \\ &= (a+b)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (c+d)\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Let $\lambda_1 = a+b = c+d$. Then you can plug in to find that $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ_1. The first eigenpair $\mathbf{A}$ is,

$$\left(\lambda_1 = a+b = c+d, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

To determine the other eigenpair $(\lambda_2, \vec{v}_2)$, we use the hint that $\vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}$. Note that by modifying the constraint $a+b=c+d$, we can also get $a-c=d-b$, which helps simplify the following:

$$\begin{aligned} \mathbf{A}\begin{bmatrix} b \\ -c \end{bmatrix} &= \begin{bmatrix} ab-bc \\ cb-dc \end{bmatrix} \\ &= \begin{bmatrix} b(a-c) \\ -c(d-b) \end{bmatrix} \\ &= (a-c)\begin{bmatrix} b \\ -c \end{bmatrix} \\ &= (d-b)\begin{bmatrix} b \\ -c \end{bmatrix} \end{aligned}$$

Therefore, we have our second eigenpair:

$$\left(\lambda_2 = a - c = d - b, \vec{v}_2 = \begin{bmatrix} b \\ -c \end{bmatrix}\right).$$

For parts (b) - (d), consider the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$$

(b) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

Solution:

From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a + b = 0.75 + 0.25 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a - c = 0.75 - 0.25 = 0.5, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

Note: If your choice of eigenvector $\vec{v}_1$ and $\vec{v}_2$ is a scaled version of the ones given in this solution, that is fine.

(c) Determine all of the *steady states* of the system. That is, find the set of points such that if Romeo and Juliet start at, or enter, any of those points, their states will stay in place forever: $\{\vec{s}_* \mid \mathbf{A}\vec{s}_* = \vec{s}_*\}$.

Solution: Any $\vec{s}_* \in \text{span}\{\vec{v}_1\}$, where $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, is the eigenvector which corresponds to the steady state, because $\vec{v}_1$ corresponds to the eigenvalue $\lambda_1 = 1$.

(d) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution:

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_2\}$. Therefore,

$$\begin{aligned} \vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_2\vec{v}_2 \end{aligned}$$

where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_2$.

If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_2^n\vec{v}_2 \end{aligned}$$

In this case $\lambda_2 = 0.5$. This means that as $n \to \infty$, $\lambda_2^n \to 0$.

Therefore,

$$\begin{aligned}\vec{s}[n] &= \alpha\lambda_2^n\vec{v}_2 \\ &= \alpha\cdot 0\cdot\vec{v}_2 \\ &= \vec{0}\end{aligned}$$

which means that

$$\lim_{n\to\infty}(R[n],J[n]) = (0,0)$$

So, ultimately, Romeo and Juliet will become neutral to each other.

Now suppose we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

Use this state-transition matrix for parts (e) - (g).

(e) Determine the eigenpairs (i.e. $(\lambda_1,\vec{v}_1)$ and $(\lambda_2,\vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.

Solution: From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a+b = 1+1 = 2, \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a-c = 1-1 = 0, \vec{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}\right).$$

(f) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n\to\infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 0$. Thus, $\vec{s}[1] = \vec{0}$. The state will remain at $\vec{0}$ for all subsequent time steps, i.e.

$$\vec{s}[n] = \vec{0}, n \geq 1$$

Therefore, Romeo and Juliet become neutral towards each other in the long run, i.e.

$$\lim_{n\to\infty}(R[n],J[n]) = (0,0)$$

(g) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n\to\infty$?

Solution:

We note that $\vec{s}[0] \in \text{span}\{\vec{v}_1\}$. Therefore,

$$\begin{aligned}\vec{s}[1] &= \mathbf{A}\vec{s}[0] \\ &= \alpha\lambda_1\vec{v}_1\end{aligned}$$

where α is the scalar that expresses $\vec{s}[0]$ as a scaled version of $\vec{v}_1$.
If we continue to apply the state transition matrix, we will see that for this $\vec{s}[0]$,

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0] \\ &= \alpha\lambda_1^n\vec{v}_1\end{aligned}$$

In this problem, $\lambda_1 = 2$. Therefore,

$$\vec{s}[n] = \alpha 2^n \vec{v}_1$$

This means that as $n \to \infty$, $\lambda_1^n \to \infty$. Essentially, the elements of the state vector continue to double at each time step and grow without bound to either $+\infty$ or $-\infty$.
Therefore, what happens to Romeo and Juliet depends on $\vec{s}[0]$. If $\vec{s}[0]$ is in the first quadrant, Romeo and Juliet will become "infinitely" in love with each other. On the other hand, if $\vec{s}[0]$ is in the third quadrant, then Romeo and Juliet will have "infinite" hatred for each other.

Finally, we consider the case where we have the following state-transition matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$$

Use this state-transition matrix for parts (h) - (j).

(h) Determine the eigenpairs (i.e. $(\lambda_1, \vec{v}_1)$ and $(\lambda_2, \vec{v}_2)$) for this system. Note that this matrix is a special case of the matrix explored in part (a), so you can use results from that part to help you.
Solution: From the results of part (a), we know that the eigenpairs of this matrix are

$$\left(\lambda_1 = a + b = 1 - 2 = -1, \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

and

$$\left(\lambda_2 = a - c = 1 - (-2) = 3, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

(i) Suppose Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$. What happens to their relationship over time if $R[0] > 0$ and $J[0] < 0$? What about if $R[0] < 0$ and $J[0] > 0$? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?
Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_2$, which has eigenvalue $\lambda_2 = 3$. Using similar methods to the solutions in part (d) and part (g), we can see that (for a given scalar α):

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0]\\ &= \alpha\lambda_2^n\vec{v}_2\\ &= \alpha 3^n\vec{v}_2\end{aligned}$$

There are two cases of long-term behavior.

Suppose, initially, that $R[0] > 0$ and $J[0] < 0$ (corresponding to $\alpha > 0$). Then as $n \to \infty$, $R[n] \to \infty$ and $J[n] \to -\infty$. Romeo will have "infinite" love for Juliet, while Juliet will have "infinite" hatred for Romeo.

Conversely, if initially $R[0] < 0$ and $J[0] > 0$ (corresponding to $\alpha < 0$), then as $n \to \infty$, $R[n] \to -\infty$ and $J[n] \to \infty$. Now Romeo would have "infinite" hatred for Juliet, while Juliet would have "infinite" love for Romeo.

(j) Now suppose that Romeo and Juliet start from an initial state $\vec{s}[0] \in \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. What happens to their relationship over time? Specifically, what is $\vec{s}[n]$ as $n \to \infty$?

Solution: The initial state $\vec{s}[0]$ lies in the span of the eigenvector $\vec{v}_1$, which has eigenvalue $\lambda_1 = -1$. As with parts (d), (g), and (i), we can see that (for a given scalar α):

$$\begin{aligned}\vec{s}[n] &= \mathbf{A}^n\vec{s}[0]\\ &= \alpha\lambda_1^n\vec{v}_1\\ &= \alpha(-1)^n\vec{v}_1\end{aligned}$$

The elements of the state vector continue to switch signs at each time step, while keeping the same magnitude.

Essentially, Romeo and Juliet maintain the same intensity (i.e. absolute value or magnitude) of feeling, but they keep changing their mind about whether that feeling is like or dislike at each time step. Note that $R[0]$ and $J[0]$ have the same sign, so they both either like each other or dislike each other at a given time step n.

2. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.

3. Midterm Problem 3

Redo Midterm Problem 3.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

(c) **Solution:** See midterm solutions.

4. Midterm Problem 4
Redo Midterm Problem 4.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

5. Midterm Problem 5
Redo Midterm Problem 5.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

(c) **Solution:** See midterm solutions.

(d) **Solution:** See midterm solutions.

6. Midterm Problem 6
Redo Midterm Problem 6.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

(c) **Solution:** See midterm solutions.

(d) **Solution:** See midterm solutions.

7. Midterm Problem 7
Redo Midterm Problem 7.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

(c) **Solution:** See midterm solutions.

8. Midterm Problem 8
Redo Midterm Problem 8.

(a) **Solution:** See midterm solutions.

(b) **Solution:** See midterm solutions.

(c) **Solution:** See midterm solutions.