EECS 16A Designing Information Devices and Systems I
Spring 2019 Homework 6

This homework is due March 8, 2019, at 23:59.
Self-grades are due March 12, 2019, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw6.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned)

Submit the file to the appropriate assignment on Gradescope.

1. Mechanical Diagonalization

All calculations in this problem are intended to be done by hand, but you can use a computer to check your work.

(a) Diagonalize the matrices A and B, i.e. compute P_A, P_A^{-1}, D_A, P_B, P_B^{-1}, and D_B such that $A = P_A D_A P_A^{-1}$ and $B = P_B D_B P_B^{-1}$ and the D matrices are diagonal with the eigenvalues along the diagonal.

$$A = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \tag{1}$$

given that A has eigenvalues $\{1,\ 2\}$ and B has eigenvalues $\{1,\ -1\}$

Solution: We want to write A as $A = P_A D_A P_A^{-1}$ where D_A is diagonal. We first compute the eigenvectors of A. For eigenvalue $= 1$ we get

$$\text{Nullspace of } (A - I) = \text{Nullspace of } \left(\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \text{Nullspace of } \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{2}$$

and for eigenvalue $= 2$ we get

$$\text{Nullspace of } (A - 2I) = \text{Nullspace of } \left(\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right) = \text{Nullspace of } \left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{3}$$

P_A is a matrix whose columns are the eigenvectors of A. (Note: the norm of the columns of P_A doesn't matter. Why not?) Thus we can write P_A and P_A^{-1} as

$$P_A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad P_A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{4}$$

Thus we can write A in diagonal form and check it

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \tag{5}$$

For B we can compute the eigenvectors.
For eigenvalue $= 1$,

$$\text{Nullspace of } (B-I) = \text{Nullspace of } \left(\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Nullspace of } \left(\begin{bmatrix} 0 & 0 \\ -2 & -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{6}$$

For eigenvalue $= -1$,

$$\text{Nullspace of } (B+I) = \text{Nullspace of } \left(\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \text{Nullspace of } \left(\begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7}$$

Again, P_B is a matrix whose columns are the eigenvectors of A. Thus we can write P_B and P_B^{-1} as

$$P_B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \qquad P_B^{-1} = \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{8}$$

Thus we can write B in diagonal form and check it

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \tag{9}$$

(b) Diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}$$

given that $\mathbf{A}$ has eigenvalues 1, 2, and 0.

Solution:

First, we compute the eigenvectors of $\mathbf{A}$ given the eigenvalues.
For the eigenvalue of 2, the eigenvector spans

$$\text{Nullspace of } \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) = \text{Nullspace of } \left(\begin{bmatrix} -\frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & -1 \end{bmatrix}\right)$$

By inspection, we get that the eigenvector is $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$. (If we didn't notice that the sum of the last two columns is 0, we could have solved it using Gaussian Elimination.)
For the eigenvalue of 1, the eigenvector spans

$$\text{Nullspace of } \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \text{Nullspace of } \left(\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}\right)$$

By inspection again, we get that the eigenvector is $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$.
For the eigenvalue of 0, the eigenvector simply spans the nullspace of $\mathbf{A}$.

$$\text{Nullspace of } \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ -1 & 1 & 1 \end{bmatrix}\right)$$

Again by inspection, we get the eigenvector is $\begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$.
We now write a matrix $\mathbf{P}$ whose columns are the eigenvectors of $\mathbf{A}$.

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We compute $\mathbf{P}^{-1}$ using Gaussian Elimination

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \underbrace{\Rightarrow}_{\substack{\text{Switching} \\ \text{row order}}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \underbrace{\Rightarrow}_{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\underbrace{\Rightarrow}_{\substack{R_3 \leftarrow R_3 - R_2 \\ R_3 \leftarrow \frac{1}{2} R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \underbrace{\Rightarrow}_{\substack{R_1 \leftarrow R_1 - R_3 \\ R_2 \leftarrow R_2 + R_3}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Thus

$$\mathbf{P}^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

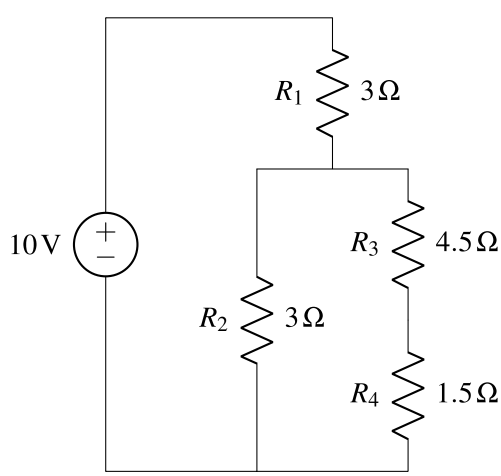
(It's a good idea to check that $\mathbf{P}\mathbf{P}^{-1} = \mathbf{I}$, which they do.)
We can now write $\mathbf{A}$ in it's diagonal form as

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(It's a good idea to multiply it out to check that you get $\mathbf{A}$.)

2. Mechanical Circuits

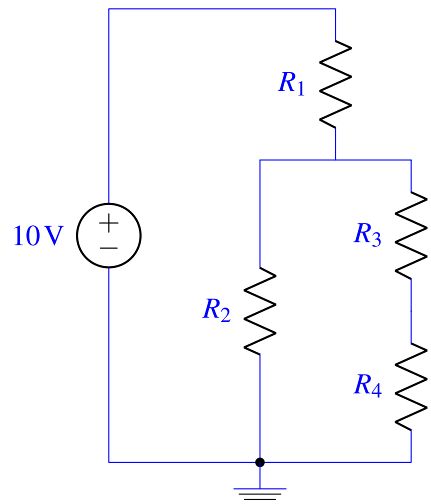
Find the voltages across and currents flowing through all of the resistors.



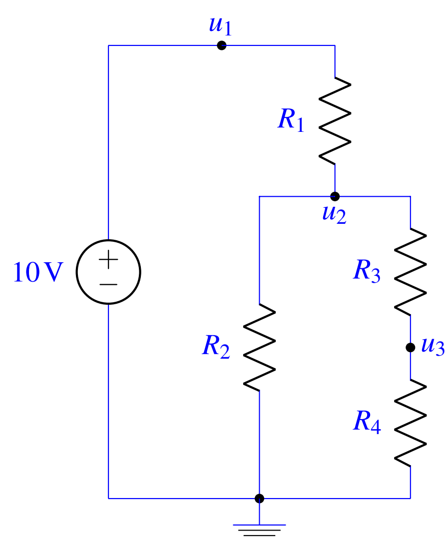
Solution:

Seven Step Method:

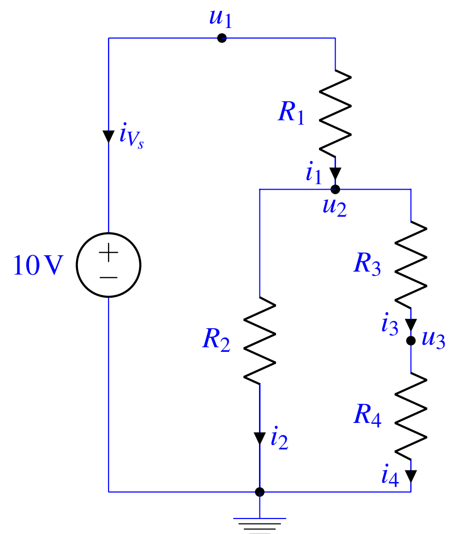
Step 1) Select a ground node. Any choice is valid, but we choose the bottom node:



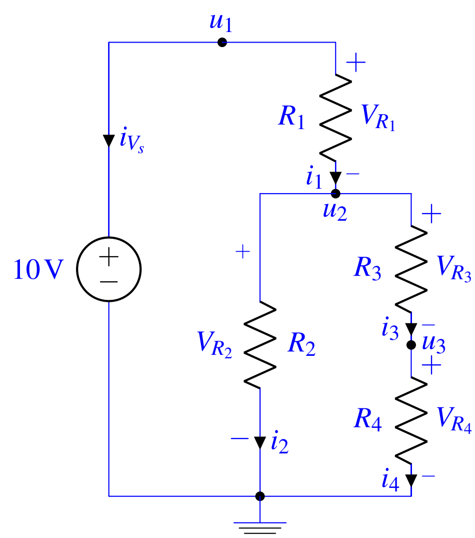
Step 2) Label all remaining nodes:



Step 3) Label the current through every **non-wire** element. The directions are arbitrary, as is what side you draw it on:



Step 4) Label the potential differences across every **non-wire** element in accordance to the passive sign convention:



Step 5) Set up an equation in the form of $\mathbf{A}\vec{x} = \vec{b}$. We have 8 unknows and so we expect a 8x8 matrix:

$$\begin{bmatrix} ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_{V_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Step 6) Given n nodes, (including ground) use KCL on $n-1$ nodes to fill in $n-1$ rows of the above system. Since we only need $n-1$ node equations, we will ignore ground. The definition of KCL is that the sum all

all currents entering and leaving the node must equal 0. We will call current leaving positive, and current entering negative. Going node by node from u_1 to u_3 in order, we set up our KCL expressions:

$$\begin{aligned} i_1 + i_{V_s} &= 0 \\ -i_1 + i_2 + i_3 &= 0 \\ -i_3 + i_4 &= 0 \end{aligned}$$

So our new system is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_{V_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

Step 7) Use IV relationships to fill in the remaining rows. Remember that only potential (voltage) *differences* make sense physically, thus we will look at the differences in node potentials and use the appropriate IV relationship to try and get useful equations.

The voltage source is connected to u_1 and ground. We know that the value of the voltage source $V_s = 10$, and that a voltage source forces the difference between its nodes to be V_s. Since we know the ground node is 0 by definition, we get

$$u_1 - 0 = V_s = 10$$

Continuing in a similar manner looking at the differences of node potentials. u_1 and u_2 are separated by a resistor, and Ohm's law relates the potential *difference* between each side of the resistor to the current through it, so we have:

$$u_1 - u_2 = i_1 R_1$$

Similarly for the other nodes

$$\begin{aligned} u_2 - 0 &= i_2 R_2 \\ u_2 - u_3 &= i_3 R_3 \\ u_3 - 0 &= i_4 R_4 \end{aligned}$$

We now have 5 more equations for our matrix, which is enough. Rearranging, we have

$$\begin{aligned} u_1 &= 10 \\ u_1 - u_2 - i_1 R_1 &= 0 \\ u_2 - 0 - i_2 R_2 &= 0 \\ u_2 - u_3 - i_3 R_3 &= 0 \\ u_3 - 0 - i_4 R_4 &= 0 \end{aligned}$$

We now have enough equations to solve. Filling in our matrix, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -R_1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_{V_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Plugging in values for the resistors and using numpy to solve this system gives:

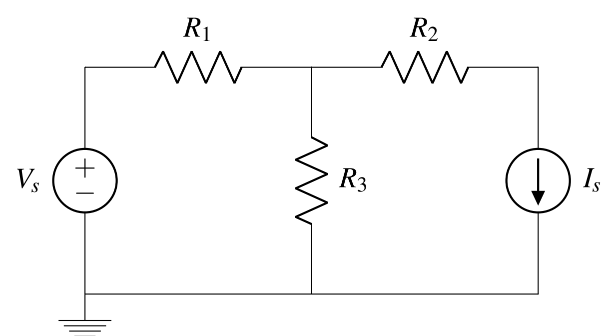
$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_{V_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{4}{3} \\ \frac{2}{3} \\ \frac{2}{3} \\ -2 \\ 10 \\ 4 \\ 1 \end{bmatrix}$$

Note a couple things. One, $i_3 = i_4$ which is expected. Since u_3 is just a single node connecting only two components, the current leaving R_3 should exactly enter R_4. Secondly, i_{V_s} is negative. This does not mean the answer is wrong, it just means that the current actually moves in the opposite direction as drawn. Don't erase your arrow and flip it if this happens! Just leave it as is and keep the negative. This physically makes sense because we have the same situation as i_3 and i_4. Both are connected to the same node and shouldn't be any different. That implies that i_{V_s} would be equal to i_1 if the arrow was flipped.

3. Circuit Analysis

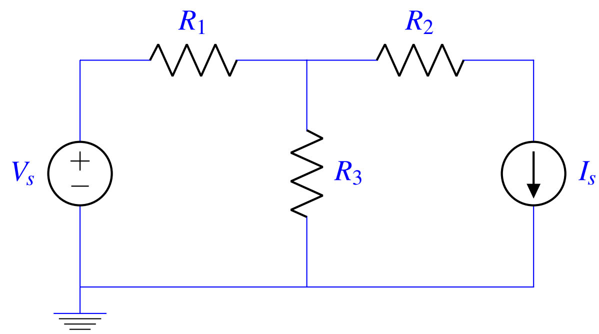
Using the steps outlined in lecture, solve the following circuits for the currents through each branch and the voltages at each node. Use the ground node labelled for you. You may use a numerical tool, such as IPython.

(a) $V_s = 5\,\text{V}, I_s = 2\,\text{A}, R_1 = R_2 = 2\,\Omega, R_3 = 4\,\Omega$

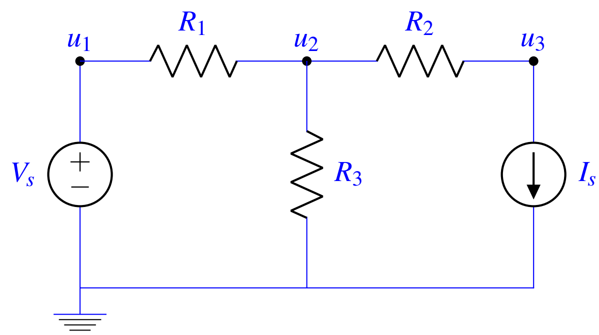


Solution:

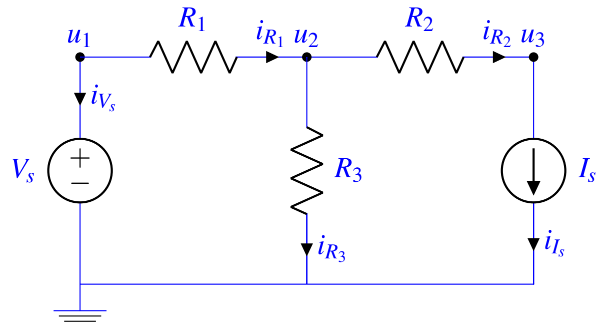
Step 1) Select a ground node.



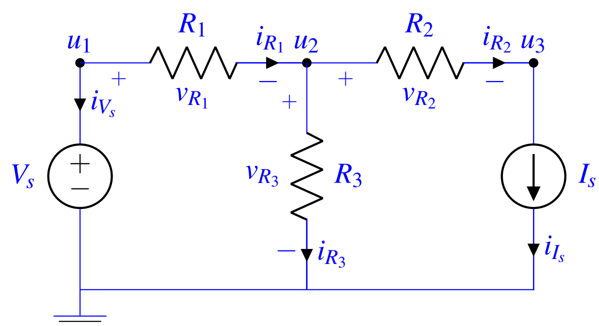
Step 2) Label all of the node potentials.



Step 3) Label all of the branch currents.



Step 4) Label all of the voltages across the elements according to passive sign convention.



Step 5) We have 8 unknowns in this system: $u_1, u_2, u_3, i_{V_s}, i_{I_s}, i_{R_1}, i_{R_2}, i_{R_3}$.
Therefore, our vector of unknowns is of size 8, and the matrix will be 8×8.

Step 6) Using KCL, we get the following equations:

$$\begin{aligned} i_{V_s} + i_{R_1} &= 0 \\ -i_{R_1} + i_{R_2} + i_{R_3} &= 0 \\ -i_{R_2} + i_{I_s} &= 0 \end{aligned}$$

Step 7) From the IV relations for all of the elements, we find the following equations:

$$\begin{aligned} u_1 - 0 &= V_s \\ i_{I_s} &= I_s \\ u_1 - u_2 - R_1 i_{R_1} &= 0 \\ u_2 - u_3 - R_2 i_{R_2} &= 0 \\ u_2 - 0 - R_3 i_{R_3} &= 0 \end{aligned}$$

Note that we now have 8 equation for 8 unknowns. Thus, we set up the following matrix relation:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -R_1 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -R_2 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -R_3 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{I_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ I_s \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

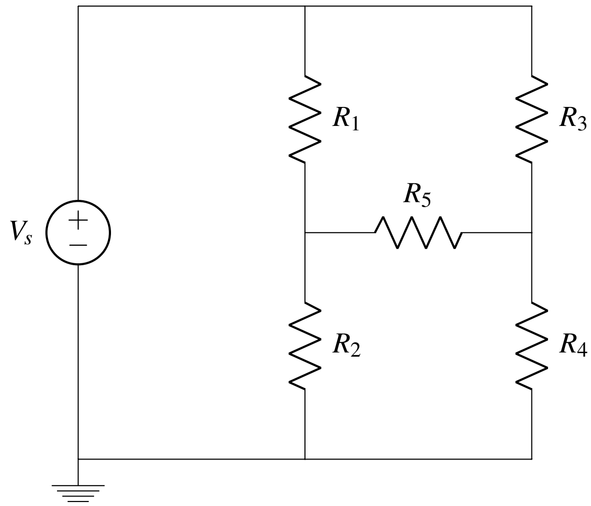
Finally, we plug in the values we were given into the matrix above and use Gaussian elimination to find the vector of unknowns.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -4 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{I_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

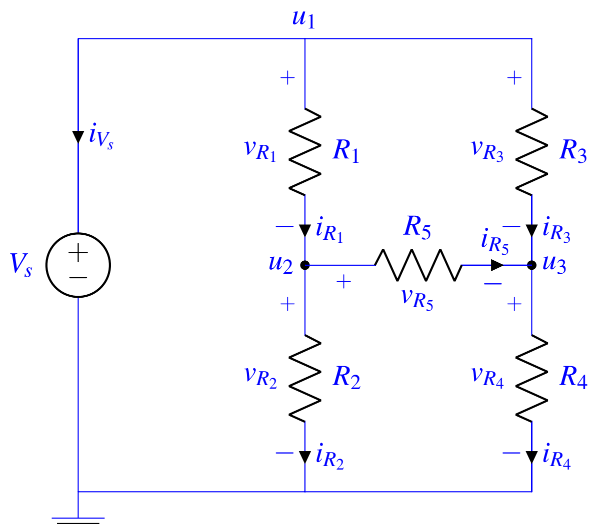
We find that:

$$\begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{I_s} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2.167 \\ 2.167 \\ 2 \\ 0.167 \\ 2 \\ 5 \\ 0.667 \\ -3.33 \end{bmatrix}$$

(b) $V_s = 5\,\text{V}, R_1 = 1\,\Omega, R_2 = 2\,\Omega, R_3 = 3\,\Omega, R_4 = 4\,\Omega, R_5 = 5\,\Omega$



Solution:
Here, we will skip showing all of the individual steps. Below is the circuit with our choice of ground and current directions.



From the above circuit, we get the following KCL equations:

$$
\begin{aligned}
i_{V_s} + i_{R_1} + i_{R_3} &= 0 \\
-i_{R_1} + i_{R_2} + i_{R_5} &= 0 \\
-i_{R_3} + i_{R_4} - i_{R_5} &= 0
\end{aligned}
$$

Using the *IV* relations for each element, we find 6 more equations:

$$
\begin{aligned}
u_1 - 0 &= V_s \\
u_1 - u_2 - i_{R_1} R_1 &= 0 \\
u_2 - 0 - i_{R_2} R_2 &= 0 \\
u_1 - u_3 - i_{R_3} R_3 &= 0 \\
u_3 - 0 - i_{R_4} R_4 &= 0 \\
u_2 - u_3 - i_{R_5} R_5 &= 0
\end{aligned}
$$

Note that we now have 9 equation for 9 unknowns. Thus, we set up the following matrix relation:

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -R_1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -R_3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -R_4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -R_5 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{R_4} \\ i_{R_5} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, we plug in the values we were given into the matrix above and use Gaussian elimination to find the vector of unknowns.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{R_4} \\ i_{R_5} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i_{V_s} \\ i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{R_4} \\ i_{R_5} \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -2.38 \\ 1.709 \\ 1.645 \\ 0.677 \\ 0.741 \\ 0.0645 \\ 5 \\ 3.29 \\ 2.968 \end{bmatrix}$$

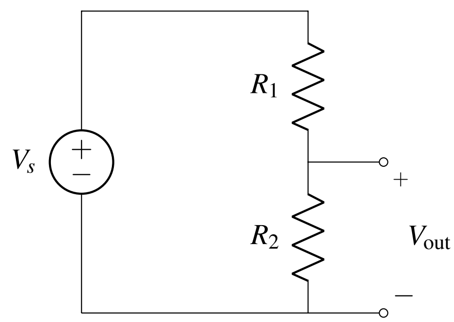
4. Temperature Sensor

Measuring quantities in the physical world is the job of sensors. This means somehow extracting that information from the world and then converting it into a form that can be observed and processed. Electrical circuits can be very useful for doing this.

For most materials, resistance increases with increasing temperature; that is, a resistor has higher resistance when it is hot than when it is cold. This is often an annoyance to circuit designers who want their circuits to work the same way at different temperatures, but this fact can also be useful. It allows us to convert temperature, a "physical" quantity, into resistance, an "electrical" quantity, to build an electronic thermometer.

In this problem, we are going to explore how effective a particular circuit made out of various types of resistors is at allowing us to measure temperature.

(a) Let's begin by analyzing a common topology, the voltage divider shown below. Find an equation for the voltage V_{out} in terms of R_1, R_2, and V_s.

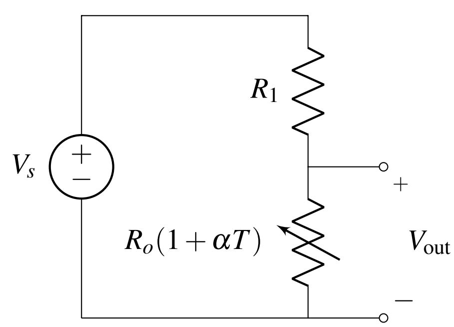


Solution:

We recognize that this circuit is a voltage divider, we can directly write:

$$V_{\text{out}} = \frac{R_2}{R_1 + R_2} V_s$$

(b) Now let's suppose that R_1 is an ideal resistor that does not depend on temperature, but R_2 is a temperature-dependent resistor whose resistance R is set by $R = R_o(1 + \alpha T)$, where T is the absolute temperature. Find an equation for the temperature T in terms of the voltage V_{out}, V_s, R_1, R_o, and α.



Solution:

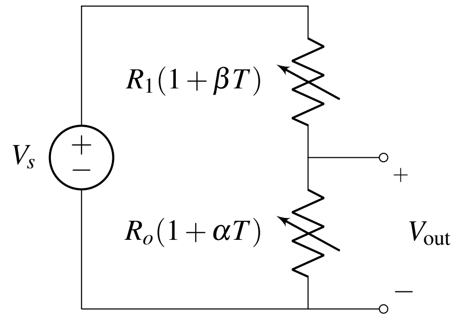
Using the relationship from the earlier part:

$$V_{\text{out}} = \frac{R_o(1 + \alpha T)}{R_1 + R_o(1 + \alpha T)} V_s$$

$$R_1 V_{\text{out}} + R_o V_{\text{out}} + R_o \alpha T V_{\text{out}} = R_o V_s + R_o \alpha T V_s$$

$$T = \frac{(R_1 + R_o)V_{\text{out}} - R_o V_s}{R_o \alpha (V_s - V_{\text{out}})}$$

(c) It turns out that almost all resisitors have some temperature dependence. Consider the same circuit as before, but now, R_1' has a temperature dependence given by $R_1' = R_1(1 + \beta T)$. Find an equation for the temperature T in terms of the voltage V_{out}, R_1, R_o, V_s, α, and β.



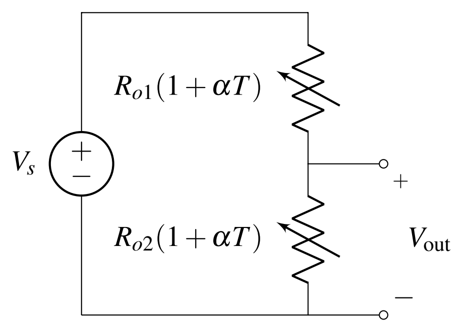
Solution:
Once again using the equation for the voltage divider:

$$V_{\text{out}} = \frac{R_o(1+\alpha T)}{R_1(1+\beta T)+R_o(1+\alpha T)} V_s$$

$$R_1 V_{\text{out}} + R_1 \beta T V_{\text{out}} + R_o V_{\text{out}} + R_o \alpha T V_{\text{out}} = R_o V_s + R_o \alpha T V_s$$

$$T = \frac{(R_1+R_o)V_{\text{out}} - R_o V_s}{R_o \alpha (V_s - V_{\text{out}}) - R_1 \beta V_{\text{out}}}$$

(d) Your colleague who has not taken EE16A thinks that they can improve this circuit's ability to measure temperature by making both resistors depend on temperature in the same way. He hence came up with the circuit shown below, where both R_1 and R_2 have nominally different values, but both vary with temperature as a function of $(1+\alpha T)$. Can this circuit be used to measure temperature? Why or why not?



Solution: Using the equation for a voltage divider:

$$V_{\text{out}} = \frac{R_{o2}(1+\alpha T)}{R_{o1}(1+\alpha T)+R_{o2}(1+\alpha T)} V_s = \frac{R_{o2}}{R_{o1}+R_{o2}} V_s$$

Notice this circuit cannot be used to measure temperature because the output voltage is independent of temperature.

5. Homework Process and Study Group

Who else did you work with on this homework? List names and student ID's. (In case of homework party, you can also just describe the group.) How did you work on this homework?

Solution:

I worked on this homework with...

I first worked by myself for 2 hours, but got stuck on problem 5, so I went to office hours on...

Then I went to homework party for a few hours, where I finished the homework.