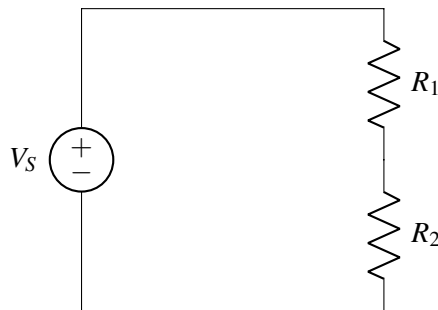


12.1 Voltage Divider

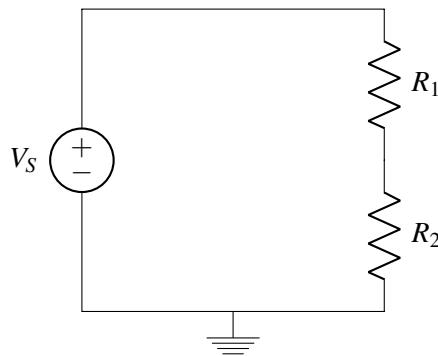
In order to review the analysis procedure described in the previous note, let's look at a particular circuit called a "voltage divider". As we will see by the end of this note, the voltage divider circuit actually lies at the heart of the resistive touchscreen that we will be studying shortly.

The voltage divider circuit consists of a voltage source (V_S) and two resistors (R_1 and R_2).

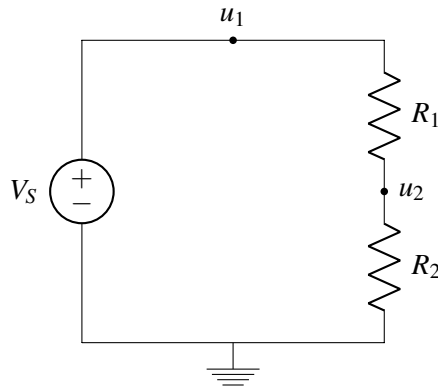


We use the circuit analysis algorithm developed in the previous note to analyze this circuit:

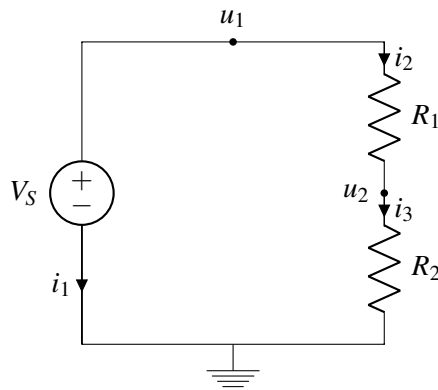
- **Step 1:** Pick a node and label it as ground.



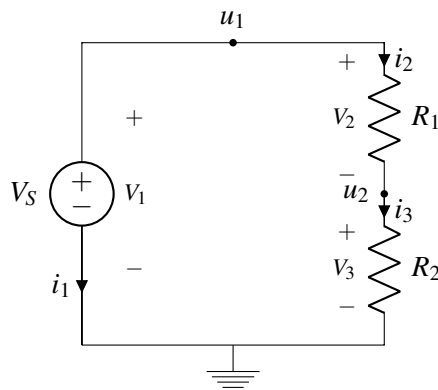
- **Step 2:** Label all remaining nodes as some u_i . There are two nodes (other than ground) in this circuit.



- **Step 3:** Label the current through every non-wire element in the circuit with i_n .



- **Step 4:** Add +/- labels on each non-wire element, following the passive sign convention. (Recall that passive sign convention means that the current enters at the positive terminal and exits are the negative terminal.)



- **Step 5:** Setup the relationship $\mathbf{A}\vec{x} = \vec{b}$ where \vec{x} consists of the unknown currents and potentials.

$$\begin{bmatrix} ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 6:** Use KCL to fill in as many linearly independent rows in \mathbf{A} and \vec{b} as possible. In this circuit, KCL gives us one equation for the node labeled u_1 and a second equation at the junction between the resistors (labeled u_2):

$$i_1 = -i_2 \rightarrow i_1 + i_2 = 0 \quad (1)$$

$$i_2 = i_3 \rightarrow i_2 - i_3 = 0 \quad (2)$$

These two equations can fill in two rows in the matrix \mathbf{A} and the vector \vec{b} :

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ? \\ ? \\ ? \end{bmatrix}$$

- **Step 7:** Use the IV relationships of each of the non-wire elements to fill in the remaining rows \mathbf{A} and \vec{b} . The IV relations give us three equations:

$$\text{Voltage Source: } u_1 - 0 = V_s. \quad (3)$$

$$\text{Resistor 1: } u_1 - u_2 = i_2 R_1 \rightarrow -i_2 R_1 + u_1 - u_2 = 0. \quad (4)$$

$$\text{Resistor 2: } u_2 - 0 = i_3 R_2 \rightarrow -i_3 R_2 + u_2 = 0. \quad (5)$$

These three equations can fill in three rows in the matrix \mathbf{A} :

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -R_1 & 0 & 1 & -1 \\ 0 & 0 & -R_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_s \\ 0 \\ 0 \end{bmatrix}$$

- **Step 8:** Solve the system of equations to determine values of unknown variables. This will result in:

$$i_1 = -\frac{V_s}{R_1 + R_2}$$

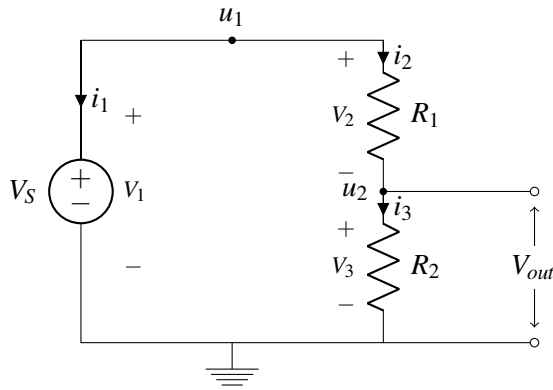
$$i_2 = \frac{V_s}{R_1 + R_2}$$

$$i_3 = \frac{V_s}{R_1 + R_2}$$

$$u_1 = V_s$$

$$u_2 = \frac{R_2}{R_1 + R_2} V_s$$

Now, we define V_{out} to be the voltage across R_2 , as shown in the circuit diagram below.



V_{out} is the same as the difference between u_2 and "0":

$$V_{out} = \frac{R_2}{R_1 + R_2} V_s = \frac{1}{1 + \frac{R_1}{R_2}} V_s$$

The reason this circuit is called a "voltage divider" is that we can create any output voltage of $V_{out} = \alpha V_s$ for any $\alpha \in [0, 1]$ (assuming that all of the resistance values are non-negative) by varying the **ratio** of the resistor values R_1/R_2 . As we will see shortly, varying this ratio is exactly the mechanism we will use to convert the relative position of a user's touch to a voltage.

12.2 Resistive Touchscreen

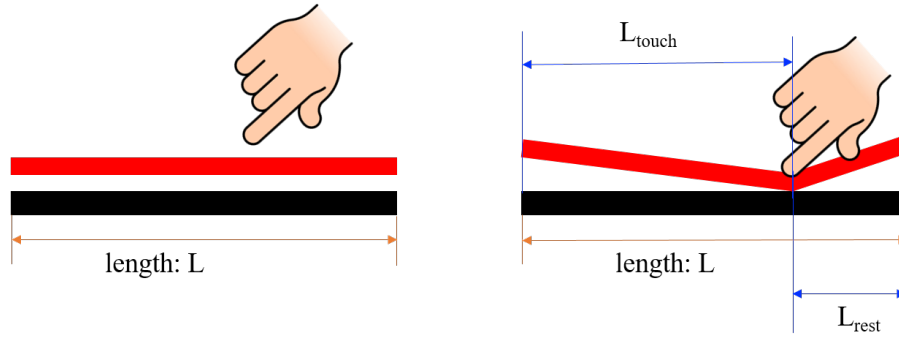
Touchscreens have revolutionized how we interact with devices. The majority of real world touchscreens are two dimensional – they find both the horizontal and vertical position of a finger press. In this note we will simplify the touchscreen to a 1D structure where we can only detect the horizontal position of the touch (or equivalently, the vertical position). In later notes we'll extend this to 2D.

Let's first examine the physical structure of a such a 1D touchscreen. The cross-section of the touchscreen is shown below. It consists of two layers of length L : the top layer (red) is flexible, and the bottom layer (black) is non-flexible. Both layers are conductive so current can flow through them. When we touch the screen, the top part contacts the bottom part at the touchpoint (in this example, we assume that the touch causes the entire width of the structure to deform uniformly and make contact to the bottom structure).

Our goal is to find the position of the touchpoint. We define L_{touch} to be the distance from the left side of the structure to the touchpoint, and L_{rest} to be the remaining distance from the touchpoint to the right side of the structure. Determining L_{touch} is equivalent to finding the position of the touchpoint.

How can we measure this physical quantity using an electrical circuit? Next we will introduce some of the physics of circuit components so that we can convert this physical structure into an electrical model. Once we have that model, we can connect additional components around it to build a complete circuit which we can then analyze (using the procedure developed in the previous note). It is worth re-emphasizing here that each of these steps is looking at different levels of abstraction – the modeling step converts a physical system into an electrical model (as we will see in this particular case, a model that is comprised of wires and resistors), and after additional components are connected to that model to create a complete circuit, we can analyze that circuit without necessarily knowing more details than are captured in the model.

Goal \rightarrow find out $L_{\text{touch}} / L = ?$



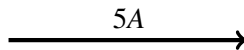
12.3 Physics of Circuits

Charge is the basic underlying quantity associated with all electrical systems. Charge can be either positive or negative, although in most materials/systems charge is carried by electrons (which are negatively charged). We measure charges with the unit **coulomb (C)**, and typically use the symbol **Q**.

Current is a measure of the movement of charge, specifically, the net amount of charge crossing through a surface in a unit time. We usually use the symbol **I** to denote current, and is defined by:

$$I = \frac{dQ}{dt}.$$

The unit for current is an **ampere (A)**, which is equivalent to 1 coulomb per second - i.e., $1C = 1A * 1sec$. We typically specify the direction of the net positive charge flow using an arrow. For example, the following arrow



represents that net +5 coulombs of charges are crossing the surface per second to the right. We could also say that net -5 coulombs of charges are crossing the surface per second to the left.

Voltage is defined as the amount of energy needed to move a unit charge between two points. We usually denote voltage with the symbol **V**. The unit associated with a voltage is a **Volt (V)**, where 1 Volt is defined such that it will require 1 Joule of energy to move 1 Coulomb (i.e. the unit of charge) between the two points (i.e. $1V = 1J/C$ or $1V * 1C = 1J$). It is important to emphasize that the definition of voltage makes it a relative quantity that can only be measured between two points; an "absolute" voltage is meaningless. As we saw in the analysis procedure, as a shorthand we will sometimes refer to a voltage at a particular point, but if we do so, we are implicitly using some other known point (often, the **ground** that we defined) as a reference against which the voltage at that single point is being measured.

A simple analogy to voltage and the fact that it is a relative quantity can be described using elevation. A mountain's summit could be 9000ft above sea level (where sea level would be the reference of an elevation of 0 ft). Alternatively, we can say that there is 9000ft between the summit and sea level. Instead, if we choose the bottom of the ocean as the reference point (elevation of 0 ft), then there will still be 9000ft between sea

level and the summit of the mountain, but the elevation at the mountain referenced to the bottom of the ocean would not be 9000ft (it would be much greater). So, just as sea level is an arbitrary (although convenient) reference level for elevation, the ground node we choose as "0V" to measure the rest of the voltage at the rest of the nodes in our circuit against is also an arbitrary reference point.

Resistance: Conductors are materials that allow current to flow through them (for example, metals such as copper). However, conductors still require some amount of work to be done to get charge (current) to move through them. This is because electrons (charge) flowing through the conductor occasionally collide with the atoms in the conductor. These collisions cause the atoms to vibrate, generating heat. Since heat is another form of energy, this process implies that energy must be spent to move the electrons through the conductor. The amount of energy that is spent (per unit charge) is described by the *resistance* of the structure.

Since voltage describes the amount of energy needed to move a charge between two points, we can capture the behavior described above by stating that when current flows through a resistor, there is also a voltage drop across that resistor. This statement is known as Ohm's law, which is:

$$V = IR$$

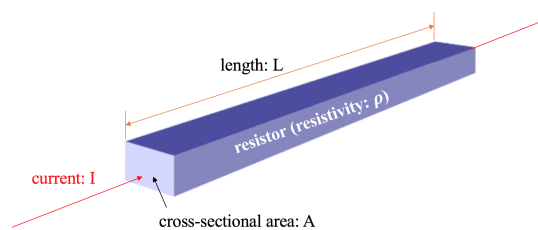
where the resistance R has units of Ohms Ω . Note that this means that $1 \Omega = \frac{1V}{1A}$.

The actual value of resistance for some physical structure is set by two things: (1) material properties, and (2) the structure's dimensions. First let's consider the material properties: every conducting material has a property known as resistivity ρ which describes how easily electrons flow through the material. For example, gold has a lower resistivity than steel. Hence, for the same physical dimensions, a gold structure will have lower resistance than a steel structure.

Next, let's consider how different dimensions impact the resistance of a structure. If we increase the length L of the structure (i.e., if we increase the dimension of the structure in the same direction as the current flows through it), this will increase the number of atoms that the electrons collide with, and hence we can expect the resistance to increase with L . If we increase the cross-sectional area A of the structure (i.e., increase the dimension in directions perpendicular to the current flow), we are allowing more electrons to flow in parallel with each other, and hence we can expect the resistance to decrease with A .

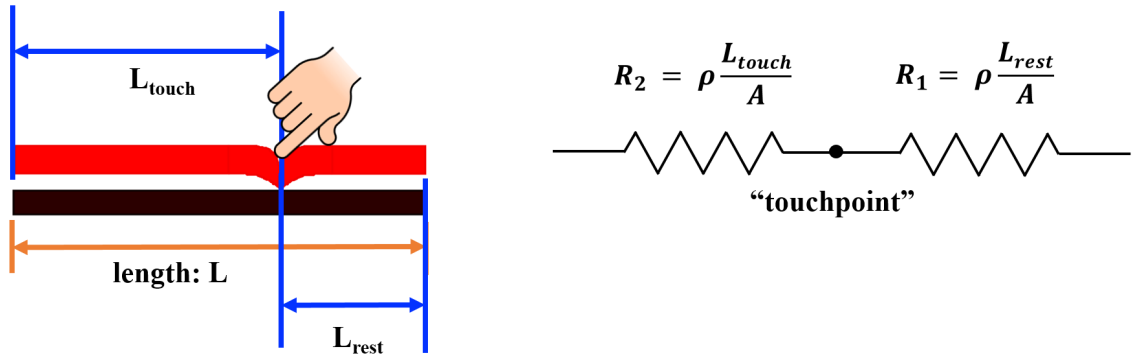
Therefore, for a physical structure made out of a material with resistivity ρ and having length L and cross-sectional area A , its resistance is given by:

$$R = \rho \times \frac{L}{A}.$$

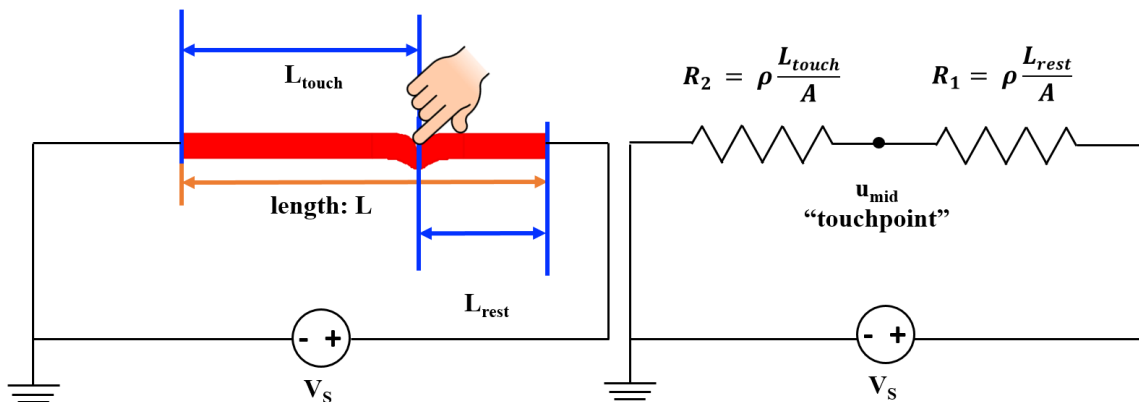


12.4 Resistive Touchscreen Revisited

Now that you've been introduced to some circuits physics, let's review the actual physical structure of the top (red) layer in the 1-D touchscreen. Since the touchpoint is where we are most interested, we can divide the top (red) layer into two parts and see what happens at the touchpoint. Given that the top (red) layer has a resistivity ρ and a cross-sectional area A , the resistance of the top layer from the touchpoint to the right-hand end is given by $R_1 = \rho \frac{L_{rest}}{A}$, the resistance of the top layer from the left-hand end to the touchpoint is given by $R_2 = \rho \frac{L_{touch}}{A}$.



We then connect a voltage source V_s to the two ends of the circuit model we just built.



This is the same circuit model that we already analyzed at the beginning of this note — it's a voltage divider circuit! From our previous analysis, we know that

$$u_{mid} = \frac{R_2}{R_1 + R_2} \times V_s. \quad (6)$$

Now we plug in our relationships between the resistances and our physical model, $R_2 = \rho \frac{L_{touch}}{A}$ and $R_1 = \rho \frac{L_{rest}}{A}$:

$$u_{mid} = \frac{\rho \frac{L_{touch}}{A}}{\rho \frac{L_{rest}}{A} + \rho \frac{L_{touch}}{A}} \times V_s. \quad (7)$$

ρ and A are shared in the denominator and the numerator so they can cancel. Then u_{mid} becomes:

$$u_{mid} = \frac{L_{touch}}{L_{touch} + L_{rest}} \times V_s. \quad (8)$$

What is $L_{touch} + L_{rest}$? It is L , the physical length of the touchscreen:

$$u_{mid} = \frac{L_{touch}}{L} \times V_s. \quad (9)$$

By connecting a voltage source to the two ends of the touchscreen and measuring u_{mid} , we can figure out the relative position of L_{touch} .

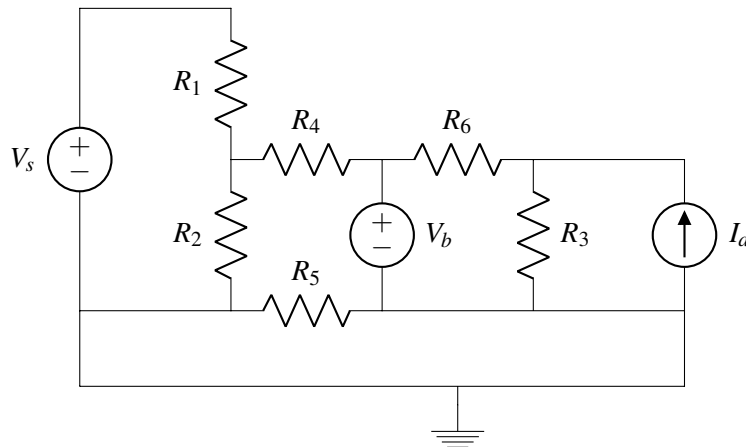
The relationship we have found between u_{mid} and V_s is very convenient because u_{mid} is not dependent on any material property such as ρ and A . This means that the top layer can be built with any material and the relationship between u_{mid} and V_s is still valid. There are always some non-idealities in the world – by making u_{mid} independent of any material property, we can make the circuit model immune to such non-idealities. We also have the freedom to choose a material for the top layer that is good for display purposes (rather than needing a specific material for the touchscreen to work).

One thing has not yet been explained – what is the function of the bottom (black) layer in the touchscreen? In the following note, we will explain how to use the bottom (black) layer to measure u_{mid} .

12.5 Practice Questions

These practice problems are also available in an interactive form on the course website.

1. How many nodes would you need to label to perform nodal analysis? Include nodes for ground and for V_s .



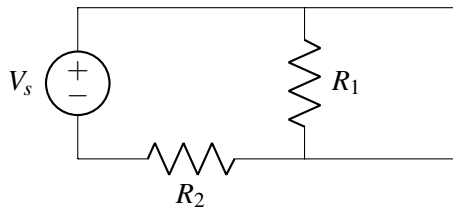
2. By what factor does the resistance of a rectangular block of metal change if you double each dimension of the block?
 - (a) 1
 - (b) 2
 - (c) $\frac{1}{2}$

(d) $\frac{1}{4}$

3. Assume that you have a cylindrical piece of metal. The cylinder has a length of 10cm and a radius of 1 mm. You measure the resistance between the two circular surfaces and get a reading of 0.54 mΩ. What is the resistivity of the metal?

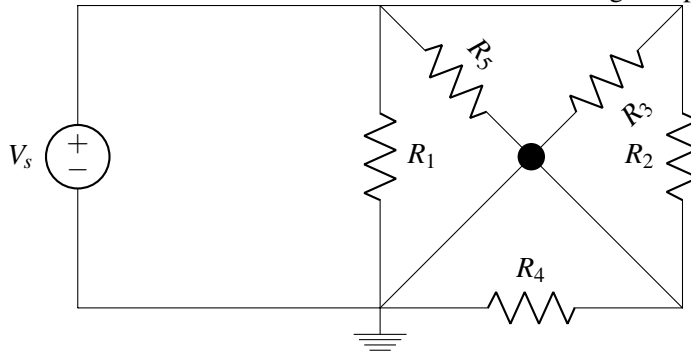
- (a) $1.7 \times 10^{-5} \Omega\text{m}$
- (b) $1.7 \times 10^{-6} \Omega\text{m}$
- (c) $1.7 \times 10^{-8} \Omega\text{m}$
- (d) $5.4 \times 10^{-9} \Omega\text{m}$

4. Find the voltage across the resistor R_1 .



- (a) V_s
- (b) $\frac{R_2}{R_1+R_2} V_s$
- (c) 0
- (d) Undefined

5. Given the circuit below, which resistors have a voltage drop equal to V_s ?



6. In the same circuit, we add a new resistor R_6 , which resistors still have a voltage drop equal to V_s ?

