EECS 16A Designing Information Devices and Systems I
Spring 2019 Lecture Notes Note 8

8.1 Subspace

In previous lecture notes, we introduced the concept of a vector space and the notion of basis and dimension. In this note, we introduce the idea of subspaces, as it is often useful to look at part of the entire set of vectors in a vector space.

Definition 8.1 (Subspace): A subspace $\mathbb{U}$ consists of a subset of the set $\mathbb{V}$ in the vector space $(\mathbb{V},\mathbb{F})$ that satisfies the following three properties:

- Contains the zero vector: $\vec{0} \in \mathbb{U}$.
- Closed under vector addition: For any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{U}$, their sum $\vec{v}_1 + \vec{v}_2$ must also be in $\mathbb{U}$.
- Closed under scalar multiplication: For any vector $\vec{v} \in \mathbb{U}$ and scalar $\alpha \in \mathbb{F}$, the product $\alpha\vec{v}$ must also be in $\mathbb{U}$.

Equivalently, a subspace is a subset of the vectors in a vector space where any linear combination of the vectors in the subset lies within the subset. Just as basis and dimension are defined for vector spaces, they have equivalent definitions for subspaces. A basis of a subspace is a set of linearly independent vectors that span the subspace, and the dimension of a subspace is the number of vectors in its bases.

In the following sections, we will explore a few key subspaces.

Additional Resources For more on subspaces, read *Strang* pages 125 - 127 and try Problem Set 3.1. In *Schaum's*, read pages 117-119 and try Problems 4.8 to 4.12, and 4.77 to 4.82.

8.2 Range

We can think of a matrix as a linear function that acts on vectors. Consider the matrix A in $\mathbb{R}^{n\times m}$ – it takes the vectors that live in $\mathbb{R}^m$ (an m-dimensional space) and outputs vectors that live in $\mathbb{R}^n$ (an n-dimensional space). We say that the **range** of an operator is the space of all outputs that the operator can map to. What is the range of our matrix operator? To answer this we write our matrix in terms of its columns,

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_m \\ | & | & & | \end{bmatrix}, \tag{1}$$

where the vectors $\vec{a}$ live in $\mathbb{R}^n$. The matrix A operates on any vector $\vec{x}$ that lives in $\mathbb{R}^m$, where the operation on $\vec{x}$ is $A\vec{x}$. Members of $\mathbb{R}^m$, $\vec{x}$ can be written as

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}. \tag{2}$$

From this we have

$$A\vec{x} = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}. = \sum_{k=1}^{m} x_k \vec{a}_k, \tag{3}$$

so we can conclude that the range of the operator A is the space of all possible linear combinations of its columns, or the *span* of (the columns of) A, which we can write as

$$span(A) = \{\vec{v} \mid \vec{v} = \sum_{i=1}^{m} x_i \vec{a}_i, \text{where } x_i\text{'s are scalars}\}, \tag{4}$$

also called the **column space** of A. We know that $range(A)$, or equivalently $span(A)$, is a subset of $\mathbb{R}^n$. However, is $range(A)$ a subspace? Let's see if it satisfies each condition described in Section 8.1.

- We know that the zero vector is in $range(A)$ because A operating on the zero vector gives the zero vector: $A\vec{0} = \vec{0}$.
- If $\vec{v}_1, \vec{v}_2$ are in $range(A)$, then there exist $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^m$ such that $A\vec{u}_1 = \vec{v}_1$ and $A\vec{u}_2 = \vec{v}_2$. Adding the two equations together, we have (due to the distributivity of matrix-vector multiplication):

  $$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A\left(\vec{u}_1 + \vec{u}_2\right). \tag{5}$$

  This tells us the $\vec{v}_1 + \vec{v}_2$ is in $range(A)$ as well.
- If $\vec{v}$ is in $range(A)$, then there exists $\vec{u} \in \mathbb{R}^m$ such that $A\vec{u} = \vec{v}$. For any scalar α, we can write

  $$\alpha\vec{v} = \alpha A\vec{u} = A\left(\alpha\vec{u}\right). \tag{6}$$

  This says that $\alpha\vec{v}$ is also in $range(A)$.

As a result, we can see that $range(A)$ is a subspace.

Additional Resources For more on column space, read *Strang* pages 127 - 129. For additional practice with these ideas, try Problem Set 3.1.

8.2.1 Rank: Dimension of the Range

What is the dimension of $range(A)$? A reasonable guess would be n, since the vectors in our span live in $\mathbb{R}^n$, making $range(A)$ a subset of $\mathbb{R}^n$ (meaning it is contained in $\mathbb{R}^n$). In general, however, the matrix operator A will not be able to produce every vector in $\mathbb{R}^n$, so our range will not equal $\mathbb{R}^n$. The dimension of $span(A)$ cannot be greater than n, since $range(A)$ is a subset of $\mathbb{R}^n$, but it can certainly be less. For example, say that A is a zero matrix. In that case, its output would be zero-dimensional, as it can only output $\vec{0}$. Recall that the dimension of a space is the minimum number of parameters needed to describe a vector in that space. Therefore, if the space only contains one vector (such as $\vec{0}$), no parameters are needed to distinguish that vector from any other vector in that space — there are no other vectors. Hence, the dimension of $range(A)$ where A is a zero matrix is just zero.

Considering the definition in Equation (4), we see that only m parameters are chosen: $x_1, x_2, \cdots, x_m$. As a result, the dimension of the span cannot be greater than m, even if m is less than n. How can this be, when the vectors in $range(A)$ each have n components? In defining our span, we have constrained the kinds of vectors that can live in our space. Therefore, we may be able to use fewer parameters than components in each vector to distinguish the vectors in this space.

Given our discussion thus far, we might be tempted to say that the span is $\min(m,n)$ — the minimum of m and n — but this is not completely true. The columns of A may be linearly dependent, meaning that some vectors are actually redundant. Any vector in $range(A)$ can always be represented as a linear combination of the linearly *independent* columns of A. For example, take

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 2 & 5 \\ 5 & 1 & 6 \\ 2 & 2 & 4 \end{bmatrix} \tag{7}$$

The last column is not linearly independent, as it can be obtained by adding the first two columns. Let us take the following linear combination of the columns of A:

$$2\begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix} \tag{8}$$

You can verify that we can obtain the same result by taking a linear combination of only the first two columns:

$$2\begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \\ 37 \\ 26 \end{bmatrix} = 6\begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}. \tag{9}$$

More generally, you can verify

$$x_1\begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 5 \\ 6 \\ 4 \end{bmatrix} = \tilde{x}_1\begin{bmatrix} 2 \\ 3 \\ 5 \\ 2 \end{bmatrix} + \tilde{x}_2\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \tilde{x}_1 = x_1 + x_3, \text{and } \tilde{x}_2 = x_2 + x_3, \tag{10}$$

Since the last column is the sum of the first two columns, we can add x_3 to x_1 and x_2 to obtain any possible linear combination of all columns in A. Since the dimension is given by the smallest number of parameters needed to identify any element in the space, it turns out that the dimension of $span(A)$ is equal to the number of linearly independent columns of A, which will be less than or equal to $\min(m,n)$.

$$\dim(span(A)) \leq \min(m,n). \tag{11}$$

Definition 8.2 (Rank): The rank of a matrix is the dimension of the span of its columns.

$$rank(A) = \dim(span(A))$$

For example, the rank of the matrix

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 2 & 5 \\ 5 & 1 & 6 \\ 2 & 2 & 4 \end{bmatrix} \tag{12}$$

defined previously is 2, since it has two linearly independent columns.

8.3 Loss of Dimensionality and Nullspace

In the previous example, we saw that the dimension of the output space can be smaller than the dimension of the input space. In this section, we'll explore where is the "remaining dimensionality" is going – somewhere call the **nullspace**.

Definition 8.3 (Nullspace): The nullspace of A consists of all vectors $\vec{x}$ in $\mathbb{R}^m$ such that $A\vec{x} = \vec{0}$:

$$N(A) = \{\vec{x} \mid A\vec{x} = \vec{0},\ \vec{x} \in \mathbb{R}^m\}. \tag{13}$$

The nullspace of A is the set of vectors that get mapped to zero by A.

What is the dimension of the nullspace? We know that it can be at most m, since all of the input vectors have m components. However, unless A is the zero matrix, not every input gets mapped to zero, so in general the dimension should be less than m. The question we need to ask is how many independent ways can we create the zero vector by taking linear combinations of the columns of A. Recall that

$$A\vec{x} = \sum_{k=1}^{m} x_i \vec{a}_k, \tag{14}$$

where again x_i are the free parameters. So our task is to find vectors $\vec{x}$ such that

$$\sum_{i=1}^{m} x_i \vec{a}_i = \vec{0} \tag{15}$$

First note that the only way $\sum_{i=1}^{m} x_i \vec{a}_i = \vec{0}$ (non-trivially) is if the columns of A are not all linearly independent. This holds by definition of linear independence. We can represent A in terms of its linearly independent columns and dependent columns, $\vec{a}^i$ and $\vec{a}^d$ respectively. Specifically, we can start with an empty set, and then we add columns of A into the set as long as the set stays linearly independent. Once we have no more columns that we can add, the columns in this set are $\vec{a}^i$ and the rest of the columns of A are $\vec{a}^d$. Of course, this set will be different depending on the order you consider the columns of A. However, the size of the set will always be the same, or equal to the dimension of $range(A)$ (Hint: show that this set is a basis for $range(A)$).

Assuming there are $j < m$ linearly independent columns[1],

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & & | & | & & | \\ \vec{a}^i_1 & \dots & \vec{a}^i_j & \vec{a}^d_1 & \dots & \vec{a}^d_{m-j} \\ | & & | & | & & | \end{bmatrix}. \tag{16}$$

We can then break up the summation in equation (15) into two summations one for the linearly independent columns and another for the linearly dependent columns,

$$\sum_{k=1}^{j} x^i_k \vec{a}^i_k + \sum_{k=1}^{m-j} x^d_k \vec{a}^d_k = \vec{0}, \tag{17}$$

where x^i and x^d are the parameters of $\vec{x}$ that multiply the linearly independent and dependent columns of A respectively. Rearranging a bit we get

$$\sum_{k=1}^{j} x^i_k \vec{a}^i_k = -\sum_{k=1}^{m-j} x^d_k \vec{a}^d_k \tag{18}$$

Remember we get to choose the parameters in our vector $\vec{x}$ that will satisfy (15). We know that in the total summation at least one linearly dependent vector must be multiplied by a nonzero parameter, since the linearly independent vectors alone cannot be linearly combined (non-trivially) to get $\vec{0}$. With this constraint let us then simplify our problem. We will impose that x^d_1 be nonzero, and set the other parameters multiplying linearly dependent vectors equal to zero. That is $x^d_2 = \ldots = x^d_{m-j} = 0$. Since $\vec{a}^d_1$ is linearly dependent on the $\vec{a}^i_k$'s we know that there exist a unique set of numbers $\beta^1_1, \beta^1_2, \ldots, \beta^1_j$ such that

$$\sum_{k=1}^{j} \beta^1_k \vec{a}_k = \vec{a}^d_1. \tag{19}$$

In other words there is a unique linear combination of our linearly independent vectors that equals $\vec{a}^d_1$. **As an aside, if a vector can be represented as a linear combination of linearly independent vectors then this representation is unique. You can try to prove this. Hint: Assume that two representations exists, set the two representations equal to one another, and see if the linear independence still holds.** Rearranging we get

[1] Please note that the linearly independent columns do not all have to be next to another, we just write it this way to ease the presentation. The results we show will still hold even if the linearly independent columns are not side by side.

$$\sum_{k=1}^{j} -\beta_k^1 \vec{a}_k^i + \vec{a}_1^d = \vec{0}, \tag{20}$$

which is also equal to

$$\sum_{k=1}^{j} -\beta_k^1 \vec{a}_k^i + \vec{a}_1^d + \sum_{k=2}^{m-j} 0\vec{a}_k^d = \vec{0}. \tag{21}$$

Notice that the last summation on the left hand side is equal to zero, and we only include it to more clearly show one of the vectors in the nullspace, namely

$$\vec{x} = \begin{bmatrix} -\beta_1^1 \\ -\beta_2^1 \\ \vdots \\ -\beta_j^1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{22}$$

Can we find others? Well the first thing we can do is multiply equation (19) by our free parameter x_1^d,

$$x_1^d (\sum_{k=1}^{j} \beta_k^1 \vec{a}_k) = x_1^d \vec{a}_1^d. \tag{23}$$

Similarly we can conclude

$$\sum_{k=1}^{j} -x_1^d \beta_k^1 \vec{a}_k^i + x_1^d \vec{a}_1^d + \sum_{k=2}^{m-j} 0\vec{a}_k^d = \vec{0}. \tag{24}$$

Since x_1^d is a free parameter any vector of the form

$$\vec{x} = \begin{bmatrix} -\beta_1^1 x_1^d \\ -\beta_2^1 x_1^d \\ \vdots \\ -\beta_j^1 x_1^d \\ x_1^d \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -\beta_1^1 \\ -\beta_2^1 \\ \vdots \\ -\beta_j^1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_1^d \tag{25}$$

will also be in the nullspace! Are there others? Yes. There is nothing special about choosing the parameter of the first linearly dependent vector to be the nonzero parameter. We can repeat the same procedure for each of the linearly dependent columns, to obtain new vectors in the nullspace. For example say that we set $x_1^d = x_3^d = \ldots = x_{m-j}^d = 0$, and leave x_2^d as our nonzero parameter we will find that

$$\vec{x} = \begin{bmatrix} -\beta_1^2 \\ -\beta_2^2 \\ \vdots \\ -\beta_j^2 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_2^d, \tag{26}$$

is also in the nullspace, where $\sum_{k=1}^{j} \beta_k^2 \vec{a}_k = \vec{a}_2^d$. This procedure can be done for each linearly dependent vector, for example if x_3^d is the nonzero parameter we will get

$$\vec{x} = \begin{bmatrix} -\beta_1^3 \\ -\beta_2^3 \\ \vdots \\ -\beta_j^3 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} x_3^d, \tag{27}$$

Furthermore, we can add vectors together from our nullspace together to get other vectors in the nullspace. **Aside: Try to prove this. Hint: if $\vec{x}_1$ and $\vec{x}_2$ are in the nullspace of A, what can be said about $A(\vec{x}_1 + \vec{x}_2)$?** This means that

$$\vec{x} = \begin{bmatrix} -\beta_1^1 \\ -\beta_2^1 \\ \vdots \\ -\beta_j^1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_1^d + \begin{bmatrix} -\beta_1^2 \\ -\beta_2^2 \\ \vdots \\ -\beta_j^2 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} x_2^d + \begin{bmatrix} -\beta_1^3 \\ -\beta_2^3 \\ \vdots \\ -\beta_j^3 \\ 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} x_3^d + \ldots + \begin{bmatrix} -\beta_1^{m-j} \\ -\beta_2^{m-j} \\ \vdots \\ -\beta_j^{m-j} \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} x_{m-j}^d \tag{28}$$

is also in the nullspace. Notice that once we choose the x^d parameters then the x^i parameters are fixed. This is because the x^d parameters control how much of the linearly dependent columns of A are being put into the summation, and the x^i parameters must ensure that the exact amount of linearly independent columns are included to cancel out the linearly dependent columns so that the output be zero. So we finally conclude that the dimension of the nullspace is equal to the number of our x^d parameters, which is equal to the number of linearly dependent columns of A. We will work out some examples in the next section.

The last point we would like to highlight here is that the dimension of the range of A is equal to the number of linearly independent columns, and the dimension of the nullspace of A is equal to the number of linearly dependent columns. Thus

$$m - \dim(range(A)) = \dim(N(A)), \tag{29}$$

so the loss of dimensionality from the input space to the output space shows up in the nullspace! This result is called the **rank-nullity theorem**.

> **Additional Resources** For more on nullspace and rank, read *Strang* pages 135 - 141 and try Problem Set 3.2.

8.4 Computing the Nullspace

So now we will show you how to compute the nullspace of a matrix systematically. Hopefully our analysis in the previous section will prove to be fruitful. For a vector to be in the nullspace it must weight the linearly independent columns appropriately to cancel out the weighted linearly dependent columns, and what we will show next will find all such vectors that do so.

In solving for the nullspace, we are fundamentally trying to solve the system of equations $A\vec{x} = \vec{0}$. We know that row reducing does not affect the solution of the system of equations, so we will assume that we've already performed Gaussian Elimination until we have an upper triangular matrix. If the matrix is not upper triangular then it can first be row reduced and the techniques will apply. Working with upper triangular matricies will make our lives much easier for two reasons: first, it is really easy to find the linearly independent columns (the columns with pivots), and it is easy to figure out how the linearly independent columns should be combined to cancel out the linearly dependent columns. We will work with the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{30}$$

First we identify the linearly dependent columns, which in this case could be columns 2 and 4. To be clear, the set of linearly dependent columns we chose is

$$\left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} \tag{31}$$

and the set of linearly independent columns includes columns 1, 3, and 5,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}. \tag{32}$$

Note that the choice of linearly dependent columns need not be unique. All that is needed is that any vector in the linearly dependent columns we choose can be written as a linear combination of the vectors in the set of linearly independent columns and, of course, the columns in the linearly independent set should be linearly independent.

We would want to find all possible scalars x_1, x_2, x_3, x_4, x_5 such that

$$x_2 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_1 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}. \tag{33}$$

Rather than considering all the linearly dependent vectors at once, we can consider them individually and sum up the contribution from each linearly dependent columns at the end. Let us first impose that the first linearly dependent column must have a weight of one in the summation and the other linearly dependent columns have weights of zero. After this, we will repeat this and only allow the second linearly dependent column to show up in the summation. Let's start with the first linearly dependent column, we want to find the unique weighting of the linearly independent columns so that the resulting sum cancels out the first linearly dependent column. It is easy to see that

$$1 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{34}$$

Now, as before, the equality would hold if we multiply both sides of the equation by any scalar $\alpha \in \mathbb{R}$

$$\alpha \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 2\alpha \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{35}$$

Note that essentially we can treat α as a free variable that can vary its value however we want while satisfying the above equation. Moving everything to the left hand side, we have

$$\alpha \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-2\alpha) \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0. \tag{36}$$

Now we see that any vector of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ \alpha \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \alpha$ is in the nullspace of A. Now let's set the weight of the second linearly dependent column to one and set the weight of the first linearly dependent column to zero. Similarly, we can find the unique weightings of the linearly independent columns that sum to the second linearly dependent column.

$$0 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 1 \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 3 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{37}$$

The equality would still hold if we multiply both sides of the equation by any scalar $\beta \in \mathbb{R}$,

$$0 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 3\beta \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 2\beta \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}. \tag{38}$$

Again, moving everything to the left hand side of the equation, we have

$$0 \times \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \times \begin{bmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-3\beta) \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-2\beta) \times \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \vec{0}. \tag{39}$$

We have that any vector of the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3\beta \\ 0 \\ -2\beta \\ \beta \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \beta$ is in the nullspace of A. We know that if any two vectors are in the nullspace, then their sum is also in the nullspace. Thus we can conclude that the nullspace of A is

$$N(A) = \{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \alpha + \begin{bmatrix} -3 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \beta \mid \alpha, \beta \in \mathbb{R} \}. \tag{40}$$

Notice that the dimension of the nullspace is 2 and the dimension of the range is 3.

8.5 Practice Problems

These practice problems are also available in an interactive form on the course website (http://inst.eecs.berkeley.edu/ ee16a/sp19/hw-practice).

1. Let $\vec{v}_1$ and $\vec{v}_2$ be two vectors in a set W. Suppose we know that $\vec{v}_1 + \vec{v}_2$ is not in W. Is W a subspace?

2. Performing Gaussian elimination on $\begin{bmatrix} 1 & -2 & 4 & 3 \\ -1 & 2 & 1 & 2 \end{bmatrix}$ gives $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. Find a basis for $\mathrm{Col}(\mathbf{A})$.

   (a) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right\}$

   (d) $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$

3. True or False: If an $m \times n$ matrix has pivots in every row, then $\mathrm{Col}(\mathbf{A}) = \mathbb{R}^m$.

4. Suppose $\mathbf{A}$ is an $m \times n$ matrix. What is the largest possible dimension of its null space, and what is the largest possible dimension of its column space?

   (a) Null space: m, Column space: m

   (b) Null space: n, Column space: n

   (c) Null space: n, Column space: $\min(m,n)$

   (d) Null space: m, Column space: $\max(m,n)$

5. Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -1 & 2 & 0 & 1 \\ 1 & -2 & 0 & -1 \\ 3 & 5 & 0 & 8 \end{bmatrix}$, find the dimension of the column space.

6. True or false: A square matrix in $\mathbb{R}^{n \times n}$ is invertible if and only its rank is equal to n.

7. True or False: If $\mathbf{A}$ is a square matrix, then $\mathrm{rank}(\mathbf{A}) = \mathrm{rank}(\mathbf{A}^2)$.