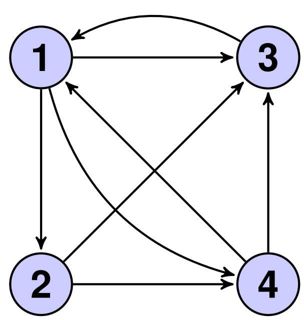
EECS 16A Designing Information Devices and Systems I
Spring 2019 Lecture Notes Note 9

9.1 Page Rank: Eigenvalues and Eigenvectors in Action

Google's Page Rank algorithm uses a web crawler to estimate the popularity of webpages. We assume that the more popular a website, the more incoming links it has. We use the following setup to model the problem: suppose we have a large population of web surfers scattered at random around the internet. At each time step, all surfers randomly choose an outgoing link on their current page to arrive at a new webpage. After a large number of time steps, we expect the number of viewers on each page to reflect that page's popularity.

It's important to realize that this is a simplified model that leaves out certain characteristics of the real world: for example, people do not click links at random and people often seek out entirely new pages rather than clicking on outgoing links. Page Rank as it's used in practice, while still a simplified model, has many more intricacies than we cover here. However, even this first order approximation is a very powerful tool.

Let's look at an internet consisting of four webpages:



How can we calculate the number of viewers on each page after infinite time steps? In this note, we will introduce three important new concepts to help us do this calculation: **Determinants**, **Eigenvalues**, and **Eigenvectors**.

Let's say we started off with 6 people on each webpage. Then one time step ticks and everyone randomly selects an outgoing link to visit. How many people do we expect to be on each webpage now?

Page 1: Page 3 only links to page 1, so all 6 people currently on 3 will move to 1. Page 4 links to pages 1 and 3, so on average about half the people on page 4 will go to page 1. There are no other incoming links, so we expect page 1 to have about $6 \times 1 + 6 \times \frac{1}{2} = 9$ people in the next time step.

Page 2: Page 1 is the only page that links to page 2. We expect a third of people from page 1 to move to page 2, so we expect page 2 to have $6 \times \frac{1}{3} = 2$ people in the next time step.

Page 3: We can arrive at page 3 from all the other pages. At page 1, the fraction of the people who move to page 3 is $\frac{1}{3}$, at page 2 the fraction of the people who move to page 3 is $\frac{1}{2}$, and at page 4 the fraction of the people who move to page 3 is $\frac{1}{2}$. Hence, we expect $6 \times \frac{1}{3} + 6 \times \frac{1}{2} + 6 \times \frac{1}{2} = 8$ people to be on page 3 in the next time step.

Page 4: We can arrive at page 4 from either page 1 or page 2. At page 1, the fraction of people who move to page 4 is $\frac{1}{3}$; at page 2, the fraction of people who move to page 4 is $\frac{1}{2}$. We expect $6 \times \frac{1}{3} + 6 \times \frac{1}{2} = 5$ people to be on page 4 in the next time step.

Notice that to find the number of people we expect to be on page i, we added up the number of people we expected would move from every other page to page i. In general, if we have n pages, $x_i(k)$ is the number of people on page i at time step k, and $p(x,y)$ is the fraction of people that will jump from page x to page y, then

$$x_i(k+1) = \sum_{j=1}^{n} x_j(k)\, p(j,i)$$

If we want to compute the expected number of viewers at time $k+1$ for all pages simultaneously, we can use matrix notation as we did with the pump and reservoir system we looked at earlier. Define $\vec{x}(k) = \begin{bmatrix} x_1(k) & x_2(k) & \dots & x_n(k) \end{bmatrix}^T$ as the vector encoding the number of viewers on each page at time k. Then

$$\vec{x}(k+1) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_n(k+1) \end{bmatrix} = \begin{bmatrix} p(1,1) & p(2,1) & \dots & p(n,1) \\ p(1,2) & p(2,2) & \dots & p(n,2) \\ \vdots & \vdots & \vdots & \vdots \\ p(1,n) & p(2,n) & \dots & p(n,n) \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$

We call the matrix of fractions P. Check that the entry $x_i(k+1)$ does in fact match the summation above. For our specific example, the matrix equation looks like this:

$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \\ 8 \\ 5 \end{bmatrix}$$

If we start with initial counts $\vec{x}(0)$ and want to find the expected number of viewers on each page at time k, we compute

$$\vec{x}(k) = P^k \vec{x}(0) = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}^k \vec{x}(0)$$

Since we only care about the relative popularity of each webpage, we can think of $\vec{x}(k)$ as fractions of viewers on each page (instead of the total number of viewers). We assume the viewers are initially distributed equally on all webpages, and we initialize $\vec{x}(0)$ as $\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^T$. After running the simulation for many time steps, we hope $\vec{x}(k)$ will reflect the popularity of the webpages. This system is only useful to us if the values of $\vec{x}(k)$ *converge* to some stable fractions.

Once our fractions exactly match these "stable fractions", running the simulation for another time step should not change the values. This means that if the fractions converge, at some point we'll reach fractions $\vec{x}^*$ such that

$$\vec{x}^* = P\vec{x}^*$$

9.2 Eigenvectors and Eigenvalues

In our Page Rank example, $\vec{x}^*$ is an example of an *eigenvector* of P. But eigenvectors have a more general definition:

Definition 9.1 (Eigenvectors and Eigenvalues): Consider a square matrix $A \in \mathbb{R}^{n \times n}$. An **eigenvector** of A is a *nonzero* vector $\vec{x} \in \mathbb{R}^n$ such that

$$A\vec{x} = \lambda \vec{x}$$

where λ is a scalar value, called the **eigenvalue** of $\vec{x}$.

How do we solve this equation for both $\vec{x}$ and λ, knowing only A? It seems we don't have enough information. We start by bringing everything over to one side:

$$A\vec{x} - \lambda \vec{x} = \vec{0}$$

We would like to factor out $\vec{x}$ on the left side, but currently the dimensions don't agree. $\vec{x}$ is an $n \times 1$ vector, while A is an $n \times n$ matrix. To fix this, we replace $\lambda \vec{x}$ with $\lambda I_n \vec{x}$, where I_n is the $n \times n$ identity matrix:

$$(A - \lambda I_n)\vec{x} = \vec{0} \tag{1}$$

Remember, in the definition of an eigenvector, we explicitly excluded the $\vec{0}$ vector. We know at least one element of $\vec{x}$ is nonzero, yet $(A - \lambda I_n)\vec{x} = \vec{0}$. To see what this means, let's define the columns of $A - \lambda I_n$ to be the vectors $v_1 \ldots v_n$ and rewrite the product in terms of the columns:

$$\begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \ldots & \vec{v_n} \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{v_1} \\ | \end{bmatrix} x_1 + \begin{bmatrix} | \\ \vec{v_2} \\ | \end{bmatrix} x_2 + \cdots + \begin{bmatrix} | \\ \vec{v_n} \\ | \end{bmatrix} x_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

From this formulation, we can see that some nonzero linear combination of the columns of $A - \lambda I_n$ results in $\vec{0}$. This means that the columns of this matrix must be linearly dependent. Now we can rephrase our problem: For what values of λ will $A - \lambda I_n$ have linearly dependent columns? To help us answer this question, we introduce the determinant.

Additional Resources For more on eigenvalues and eigenvectors, read *Strang* pages 288 - 291 or read *Schuam's* pages 296-299.

9.3 Determinants

Every square matrix has a quantity associated with it known as the *determinant*. This quantity encodes many important properties of the matrix, and it is also intimately connected to its eigenvalues.

For this class, we only need to know how to compute the determinant of a 2×2 matrix by hand. This is given by

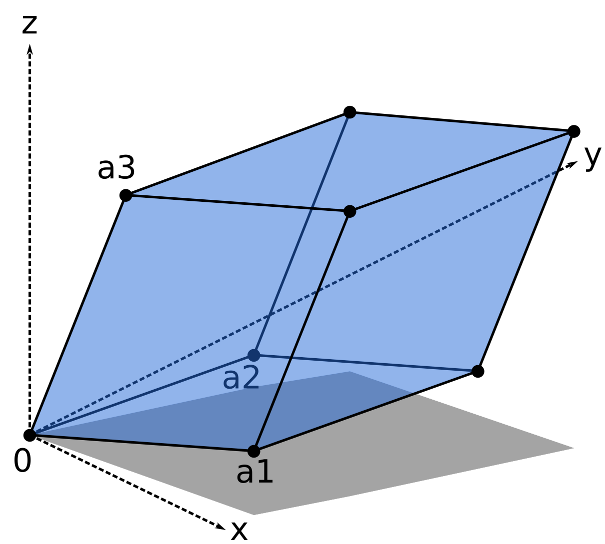
$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

.

There is a beautiful recursive definition for determinants of general $n \times n$ matrices which is outside the scope of this class, but you can read the Wikipedia article on determinants if you want to learn more about it.

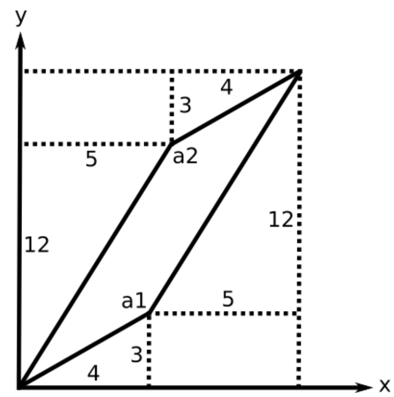
Suppose we have a $n \times n$ square matrix $A - \lambda I_n$ and a non-zero vector $x \in \mathbb{R}^n$. Recall from the previous section that the columns of $A - \lambda I_n$ are linearly dependent. We will show that if a matrix's columns are linearly dependent, then its determinant is zero.

Let's consider the geometric connection between the column vectors of a matrix and its determinant. For simplicity, first consider a 3×3 matrix. Let its column vectors be a_1, a_2 and a_3. Then consider the parallelepiped formed by these vectors: [1]



We define the determinant of this matrix as the volume of this parallelepiped formed by its column vectors. For a 2×2 matrix, its determinant is the area of the parallelogram generated by its 2 column vectors. Similarly the determinant of a 4×4 matrix is the 4-dimensional volume formed by its 4 column vectors. You can check the formula for the determinant of a 2×2 matrix by computing the area of the parallelogram below using geometry and then comparing that to the value given by the determinant formula.

[1] Modified from image by Claudio Rocchini, `https://en.wikipedia.org/wiki/Determinant`



$$\det\left(\begin{bmatrix} a_1 & a_2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 4 & 5 \\ 3 & 12 \end{bmatrix}\right) = 48 - 3 \times 5 = 33$$

So now back to our proof, so what happens to the determinant when the column vectors of the matrix $det(A - \lambda I_n) = 0$ are linearly dependent? Remember from the previous notes, this means that either one vector lies in the plane formed by the other two, or all three lie on the same line! In either case, the parallelepiped will be "compressed" into a plane or line, with zero volume. Thus the determinant (volume of the parallelepiped) has to be zero.

Additional Resources For more on determinants, read *Strang* pages 247 - 253 and pages 254 to 257, and try Problem Set 5.1.
In *Schaum's*, read pages 264-265 and pages 265-266, and try Problems 8.1, 8.38, 8.39, 8.2, and 8.40 to 8.43.

9.4 Computing Eigenvalues with Determinants

Now we can go back to finding eigenvalues and eigenvectors. Recall that we are trying to solve the equation

$$(A - \lambda I_n)\vec{x} \;=\; \vec{0}$$

Since we are only interested in solutions where $\vec{x} \neq \vec{0}$, we want to find values of λ such that $(A - \lambda I_n)$ has linearly dependent columns. We just learned that a matrix with linearly dependent columns has determinant equal to zero:

$$(A - \lambda I_n)\vec{x} \;=\; \vec{0} \implies \det(A - \lambda I_n) \;=\; 0 \tag{2}$$

Now, let's use this to find the eigenvalues of a matrix. Consider the example 2×2 matrix

$$A \;=\; \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

We now expand the expression above:

$$A - \lambda I_n = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$$

We now take the determinant of this matrix and set it equal to 0:

$$(1-\lambda)(3-\lambda) - 4 \times 2 = 0$$

Expanding this expression, we get

$$\lambda^2 - 4\lambda - 5 = (\lambda+1)(\lambda-5) = 0$$

We find that there are two solutions to this equation, and therefore two eigenvalues: $\lambda = -1, \lambda = 5$. Each eigenvalue will have its own corresponding eigenvector. To find these, we plug each value of λ into the Equation 2 and solve for $\vec{x}$. Starting with $\lambda = 5$:

$$(A - 5I_2)\vec{x} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

We see that both of the rows provide redundant information: $4x_1 - 2x_2 = 0$. The eigenvectors associated with $\lambda = 5$ are all of the form

$$\alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R}$$

Next, we will find the eigenvector associated with $\lambda = -1$. We plug this value of λ into the Equation 2 and solve:

$$(A + I_2)\vec{x} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0}$$

Both rows provide the information $x_1 = -x_2$. The eigenvectors associated with $\lambda = -1$ are of the form

$$\alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R}$$

In summary, the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has eigenvalues $\lambda = 5, -1$ with associated eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We can check by multiplying the eignenvectors with A and seeing that they get scaled by their corresponding eigenvalues.

$$\begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}5\\10\end{bmatrix} = 5\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\4 & 3\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} = -\begin{bmatrix}1\\-1\end{bmatrix}$$

Additional Resources For more on computing eigenvalues, read *Strang* pages 292 - 296. For additional practice, try Problem Set 6.1.

9.5 Repeated and complex eigenvalues

Every 2×2 matrix has two eigenvalues, but they do not have to be real or unique!

Repeated Eigenvalues:

For a 2×2 matrix, it's possible that the two eigenvalues that you end up with have the same value, leading to a phenomenon called a *Repeated Eigenvalues*. This repeated eigenvalue can one or two associated eigenvectors (unlike a single, unrepeated eigenvalue, which will only have one associated eigenvector). If there are two eigenvectors, they form an *eigenspace*, which is the space of all vectors $\vec{v}$ for which $A\vec{v} = \lambda\vec{v}$.

For example, the following matrix has a repeated eigenvalue of λ.

$$A = \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}$$

The *eigenspace* of this matrix is all of $\mathbb{R}^2$ since for any vector $\vec{v} \in \mathbb{R}^2$, $A\vec{v} = \lambda\vec{v}$.

Complex Eigenvalues:

Sometimes when we solve $\det(A - \lambda I_n) = 0$, there will be no real solutions to λ. Consider the transformation that rotates any vector by θ, about the origin:

$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$$

There are certain specific cases when R has real eigenvalues: When $\theta = 0$, the rotation matrix reduces to the identity matrix, which does not change change vectors, and therefore has eigenvalue $\lambda = 1$. When $\theta = \pi$, the rotation matrix reduces to the negative identity matrix, which changes the sign of vectors and therefore has eigenvalue $\lambda = -1$. In both these cases, all vectors in $\mathbb{R}^2$ are eigenvectors, so $\mathbb{R}^2$ is the eigenspace.

However, when θ does not correspond to 0° or 180° rotation, there are no vectors that are scaled versions of themselves after the transformation. This will results in complex eigenvalues. For example, let's look at 45° rotation:

$$R = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

$$\det(R-\lambda I)=\frac{1}{2}(1-\lambda)(1-\lambda)+\frac{1}{2}$$

Setting this determinant equal to zero and solving yields the complex eigenvalues, $\lambda=\frac{1}{\sqrt{2}}(1+i)$ and $\lambda=\frac{1}{\sqrt{2}}(1-i)$, which makes sense because there are no physical (real) eigenvectors for this transformation.

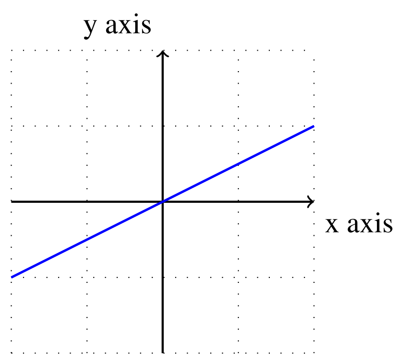
9.6 Examples

In Section 9.4, we saw an example of how to compute the eigenvalues and eigenvectors of a matrix. Here, we'll look at another example where we do the opposite – finding a matrix given it's eigenvalues and eigenvectors. Finally, we'll go back to our page rank example and rank the popularity of webpages by finding the appropriate eigenvector.

Example 9.1 (Finding a matrix from its eigenvalues and eigenvectors): Now we turn to the case where we try to derive the matrix from the provided vectors. This is essentially reverse engineering the previous problem and reworking the derivation of the eigenvalues.

We would like to solve the problem: what is the matrix A that reflect any $2D$ vectors over the vector $\begin{bmatrix}2\\1\end{bmatrix}$?

Below is a graphical representation of that vector:



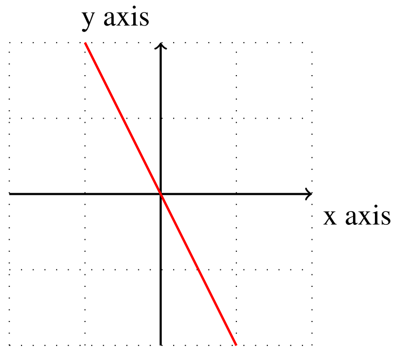
Remember that an eigenvector of a matrix is any vector that results in a scaled version of itself after being transformed by the matrix. Can you think of any eigenvectors of this reflection transformation? First we observe that anything lying on the line of reflection will be unchanged, in other words, scaled by one. Therefore, we can write the eigenvector, eigenvalue pair:

$$\lambda_1=1$$

$$\vec{v}_1=\alpha\begin{bmatrix}2\\1\end{bmatrix}$$

In addition, any vector lying perpendicular to the line of reflection will be transformed to point in the *opposite* direction, in other words, scaled by -1. This results in another eigenvector, eigenvalue pair:

$$\lambda_2=-1$$



$$\vec{v}_2 = \alpha \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Now we can use our definition, $A\vec{v} = \lambda\vec{v}$, to set up a system of equations so we can solve for matrix A:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

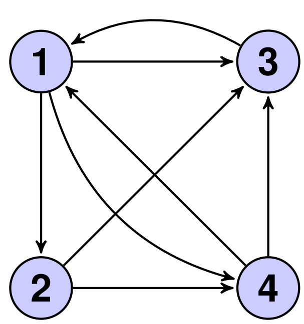
We rearrange these linear equations as follows. (*To see these are equivalent, try writing out all four equations in terms of* $a_1 \dots a_4$.)

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Finally, we solve the system of equations with Gaussian Elimination to get our transformation matrix, A.

$$A = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$$

Example 9.2 (Page Rank): Let's go back to our page rank example from the beginning of this note. Recall that we have four webpages, with links between them represented by the diagram below:



We can write the tranformation matrix P that tells us how the number of people on each webpage changes over time:

$$\vec{x}(t+1) \; = \; P\vec{x}(t) \; = \; \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \vec{x}(t)$$

where $\vec{x}(t)$ is a vector describing the fraction of people on each page at time t.

We are interested in the *steady state* vector, $\vec{x^*}$ such that

$$\vec{x^*} = P\vec{x^*}.$$

Looking back at our definition of eigenvalues and eigenvectors, we can see that $\vec{x^*}$ is the eigenvector associated with eigenvalue $\lambda = 1$. In this specific case of finding a steady state vector, we already know the eigenvalue, so proceed to finding the associated eigenvector. (More generally, we would use the determinant to find the eigenvalues first).

From Equation 1, we can write that our steady state vector satisfies

$$(P - \lambda I)\vec{x^*} = \vec{0}$$

where $\lambda = 1$.

$$\left(\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \vec{x^*} = \vec{0}$$

$$\begin{bmatrix} -1 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -1 \end{bmatrix} \vec{x^*} = \vec{0}$$

We solve for $\vec{x^*}$ by finding the null space of the above matrix. After performing Gaussian elimination on the matrix above, we get the following row equivalent system of equations:

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{x^*} = \vec{0}$$

Remember that each row represents an equation:

$$x_1 - 2x_4 = 0$$
$$x_2 - \frac{2}{3}x_4 = 0$$
$$x_3 - \frac{3}{2}x_4 = 0$$

We can represent this set of solutions as a vector where x_4 is a free variable, $x_4 \in \mathbb{R}$.

$$\vec{x^*} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_4 \\ \frac{2}{3}x_4 \\ \frac{3}{2}x_4 \\ x_4 \end{bmatrix}$$

Since we want, the steady state vector to represent the fraction of people on each page, we can choose x_4 so that the total number of people sums to 1.

$$\vec{x^*} = \frac{1}{31}\begin{bmatrix} 12 \\ 4 \\ 9 \\ 6 \end{bmatrix}$$

From our steady state solution, we conclude that webpage 1 is most popular, then webpages 3, 4, and 2.

9.7 Practice Problems

These practice problems are also available in an interactive form on the course website.

1. What are the eigenvalues of the matrix $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$?
2. What are the eigenvectors of the matrix $\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$?
3. True or False: If an $n \times n$ matrix $\mathbf{A}$ is not invertible, then it has an eigenvalue $\lambda = 0$.
4. True or False: If an invertible matrix $\mathbf{A}$ has an eigenvalue λ, then $\mathbf{A}^{-1}$ has the eigenvalue $\frac{1}{\lambda}$.
   (a) Always true.
   (b) False.
   (c) True only if $\lambda \neq 0$.
5. Two students find two different eigenvectors associated with the same eigenvalue of a matrix, $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -\frac{1}{2} \\ 2 \end{bmatrix}$. Student One insists that the other must have made an error because every eigenvalue is associated with a unique eigenvector. Student Two insists that they are both valid answers. Who is right, Student One or Student Two?
6. True or False: If for some matrix $\mathbf{A}$, $\mathbf{A}\vec{x_1} = \lambda_1\vec{x_1}$ and $\mathbf{A}\vec{x_1} = \lambda_2\vec{x_1}$, then $\lambda_1 = \lambda_2$.
7. True or False: A matrix with only real entries can have complex eigenvalues.

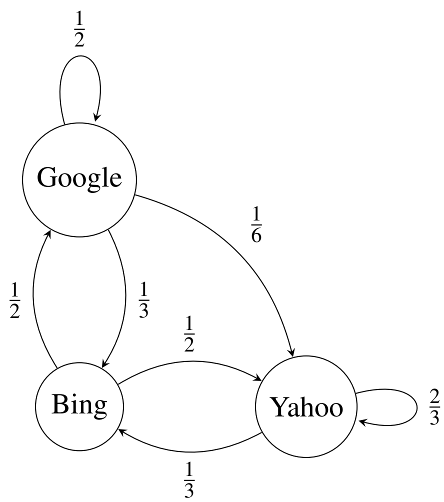
8. True or False: If $\mathbf{A}^T$ has an eigenvalue λ, then $\mathbf{A}$ also has the eigenvalue λ.

9. True or False: A diagonal $n \times n$ matrix has n distinct eigenvalues.

10. For a square non-invertible $n \times n$ matrix $\mathbf{A}$, what is the maximum and minimum number of distinct eigenvalues it can have?

    (a) Max: 1, Min: 0
    (b) Max: n, Min: 1
    (c) Max: n, Min: $n-1$
    (d) Max: $n-1$, Min: 0
    (e) Max: $n-1$, Min: 1

11. Find the steady state, if it exists, of the following system assuming that the state vector is $\vec{x}[t] = \begin{bmatrix} x_{\text{Google}}[t] \\ x_{\text{Yahoo}}[t] \\ x_{\text{Bing}}[t] \end{bmatrix}$ and that we start off with $\vec{x}[0] = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}$.



12. What is the matrix $\mathbf{A}$ that reflects any vector in $\mathbb{R}^2$ about the line through the origin in the direction of $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$?