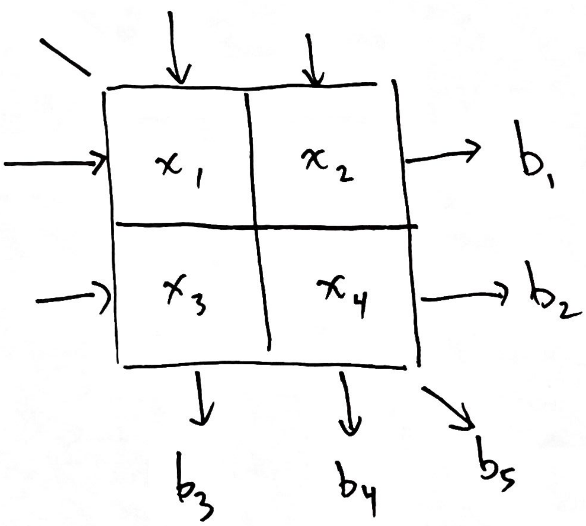
EECS 16A
Lec 0B, 1/23/20.

Tomography Example.



$$\left.\begin{aligned} x_1 + x_2 \quad\quad\quad\quad &= b_1 \\ x_3 + x_4 &= b_2 \\ x_1 + \quad\quad x_3 \quad\quad &= b_3 \\ x_2 \quad\quad + x_4 &= b_4 \\ x_1 \quad\quad\quad\quad + x_4 &= b_5 \end{aligned}\right\}$$

Assume $b_1 = b_2 = b_3 = b_4 = 1$

$x_1 = x_2 = x_3 = x_4 = 1/2$.

$x_2 = x_3 = 1 \quad x_1 = x_4 = 0$

System of Linear Equations

$e^{x_1} + \log x_2 = 5$

$\sin(x_1) + \cos(x_2) = 72$

collection of linear equations.

Linear Algebra = tools for working with linear equations.

Linear equation: $f(x_1 \cdots x_n) = b$ $\qquad b \in \mathbb{R}$

$\uparrow$ $\qquad f: \mathbb{R}^n \longrightarrow \mathbb{R}$

f = linear function.

<u>Def</u>: A function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is linear if:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \cdots \alpha x_n + \beta y_n) = \alpha f(x_1, \cdots x_n) + \beta f(y_1 \cdots y_n)$$

$\forall\ \alpha, \beta, x_1 \cdots x_n, y_1 \cdots y_n \in \mathbb{R}$.

<u>Fact</u>: Every linear function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ has the form

$$f(x_1 \cdots x_n) = a_1 x_1 + a_2 x_2 + \cdots \quad a_n x_n$$

where a_i's = coefficients (just numbers not depending on x_i's).
x_i's = variables.

Why?

$$f(x_1, \cdots x_n) = f(\underbrace{x_1 \cdot 1 + 1 \cdot 0}_{x_1}, \underbrace{x_1 \cdot 0 + 1 \cdot x_2}_{x_2}, \underbrace{x_1 \cdot 0 + 1 \cdot x_3}_{x_3}, \cdots, \underbrace{x_1 \cdot 0 + 1 \cdot x_n}_{x_n})$$

$$= x_1 f(1,0,0 \cdots 0) + f(0, x_2, x_3, \cdots x_n)$$

$$\vdots$$

$$= x_1 \underbrace{f(1,0,0 \cdots 0)}_{a_1} + x_2 \underbrace{f(0,1,0,0 \cdots 0)}_{a_2} + \cdots + x_n \underbrace{f(0,0, \cdots 0, 1)}_{a_n}$$

$$= a_1 x_1 + a_2 x_2 + \cdots a_n x_n .$$

So, a system of linear EQns. has the form:

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 \cdots &= b_2 \\
\vdots \qquad & \quad \vdots \\
a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n &= b_m
\end{aligned}$$

a_{ij}'s = coefficients (design)

x_i's = variables (unknowns)

b_i's = constants (measurements)

Shorthand notation for expressing the system of equations:

$$\begin{array}{cc} & n \text{ variables} \\ m \text{ eq}^{ns} & \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & & \vdots & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & & a_{mn} & b_m \end{array}\right] \end{array}$$

augmented matrix

Ex: Tomography

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & b_1 \\ 0 & 0 & 1 & 1 & b_2 \\ 1 & 0 & 1 & 0 & b_3 \\ 0 & 1 & 0 & 1 & b_4 \\ 1 & 0 & 0 & 1 & b_5 \end{array}\right]$$

Question: How to systematically solve Systems of LEQs?

Ex:

$$2x + 3y = 8$$
$$3x - y = 1$$

$$\begin{bmatrix} 2 & 3 & | & 8 \\ 3 & -1 & | & 1 \end{bmatrix}$$

$R_1 \leftarrow R_1/2$

$$x + 3/2\,y = 4$$
$$3x - y = 1$$

$$\begin{bmatrix} 1 & 3/2 & | & 4 \\ 3 & -1 & | & 1 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 3R_1$

$$x + 3/2\,y = 4$$
$$-11/2\,y = -11$$

$$\begin{bmatrix} 1 & 3/2 & | & 4 \\ 0 & -11/2 & | & -11 \end{bmatrix}$$

$R_2 \leftarrow -2/11\,R_2$

$$x + 3/2\,y = 4$$
$$y = 2$$

$$\begin{bmatrix} 1 & 3/2 & | & 4 \\ 0 & 1 & | & 2 \end{bmatrix}$$

$R_1 \leftarrow R_1 - 3/2\,R_2$

$$x = 1$$
$$y = 2.$$

$$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 2 \end{bmatrix}$$

Operations at each step:

- rescale a row by multiplying by non-zero scalar
- add to any row, a scalar multiple of any other row
- swap any two rows.

These are called <u>elementary row operations</u>.

<u>Gaussian Elimination</u>: algorithm for solving a system of linear equations by performing elementary row operations on the augmented matrix.

How does it go? You already did it!

<u>Def</u>: The <u>leading entry</u> of a row is the first non-zero entry in that row

$$\left[\begin{array}{cc|c} \boxed{1} & 3/2 & 4 \\ 0 & \boxed{-11/2} & -11 \end{array}\right]$$

Gaussian Elimination:

Step 1: [reduction to row-echelon form]

For $i = 1 \cdots m$

1. If necessary, swap row i with a row below it so that the leading entry in row i is as far left as possible.

2. Scale row i so that leading entry is 1.

3. For $j = i+1 \cdots m$, add to row j a scalar multiple of row i so that the leading entry in row i has all zeros below it.

Observation: Step 1 will result in an augmented matrix of the form

row-echelon form

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$*$ = anything

3 Properties:
1) All-zero rows are at bottom
2) For each leading entry (pivot), all entries to left and/or below = 0
3) All leading entries = 1.

Step 2: [back-substitution] For $i = m \dots 1$, Add to ~~row~~ each row $j = 1 \dots i-1$ a multiple of row i, so that leading entry of row i has all zeros above it

Observe: Leaves us with a matrix of form:

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Such a matrix satisfies 2 properties:

1) It is in row-echelon form
2) each leading entry is the only non-zero entry in its column

This is called reduced row-echelon form (rref)

Fact: Each augmented matrix has a unique corresponding matrix in rref.

Once augmented matrix is reduced to rref, variables corresponding to columns containing pivots (leading entries) are called <u>basic variables</u>. Other variables are called <u>free variables</u>.

<u>EX:</u>

$$2x + 3y = 8$$
$$2x + 3y = 6$$

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 2 & 3 & 6 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & 0 & -2 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1/2} \left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 0 & -2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 4 \\ 0 & 0 & 1 \end{array}\right]$$

$$x + 3/2\, y = 4$$
$$0 = 1$$

impossible $\Rightarrow$ no solution

Ex: $x + 4y = 6$

$2x + 8y = 12$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 2 & 8 & 12 \end{array}\right]$$

$R_2 \leftarrow R_2 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

basic variable   free variable

$x + 4y = 6$

$0 = 0$

Infinitely many solutions. To obtain a solution, select free variable freely

$y = 0 \quad x = 6$

If a system of LEQs has a solution: consistent

" " " " " doesnt have a solution: inconsistent.