Last time: • Intro to Module 3 (ML+)
• Inner Products

Today: More on Inner Products, with applications to Classification and Estimation. (TA: Amanda)

Recall: $\langle \cdot , \cdot \rangle$ is an inner product on real vector space $\mathcal{V}$ if:

✓1) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \quad \forall \vec{u}, \vec{v} \in \mathcal{V}$

✓2) $\langle \alpha\vec{u} + \vec{w}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$
$\forall \alpha \in \mathbb{R}, \ \vec{u}, \vec{v}, \vec{w} \in \mathcal{V}$

Ex: $\left\langle 3\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix} \right\rangle = 3\left\langle \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix} \right\rangle + \left\langle \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}4\\-1\end{bmatrix} \right\rangle$

✓3) $\langle \vec{u}, \vec{u} \rangle \geq 0 \quad \forall \vec{u} \in \mathcal{V}$
with equality only if $\vec{u} = \vec{0}$.

Induced Norm: $\boxed{\|\vec{u}\| \overset{\text{def}}{=} \sqrt{\langle \vec{u}, \vec{u} \rangle}}$

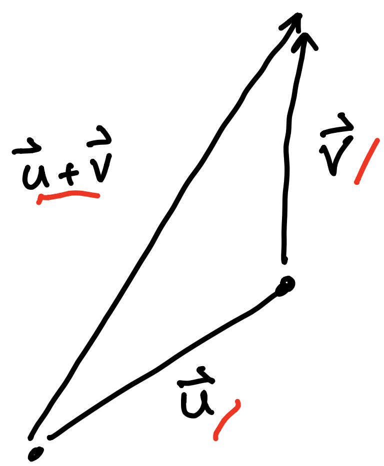
defines a norm on $\mathcal{V}$. i.e., $\|\cdot\|$ satisfies

1) $\|\alpha \vec{u}\| = |\alpha| \, \|\vec{u}\|$, 2) $\|\vec{u}\| = 0 \iff \vec{u} = \vec{0}$

3) $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$.

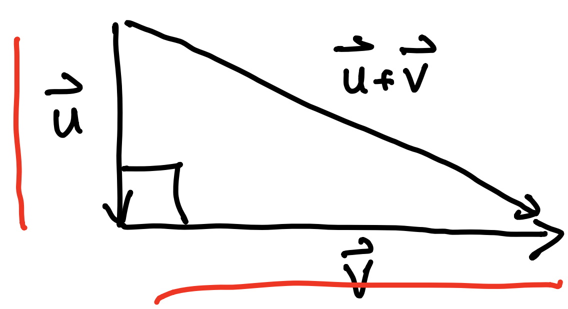
Note: Even though $\|\cdot\|$ is not necessarily Euclidean norm, most of the intuition still applies.

E.g. Triangle Inequality: $\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|$.



(Homework exercise using Cauchy-Schwarz Inequality: $|\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\| \, \|\vec{v}\|$)

E.g. Pythagorean Theorem



$\|\cdot\|$ = norm

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$

$\llcorner$ = "perpendicular" in Euclidean geometry

= "orthogonal" for general inner products

Def: For inner product $\langle \cdot , \cdot \rangle$, two vectors $\vec{x}, \vec{y}$ are said to be <u>orthogonal</u> if

$$\langle \vec{x}, \vec{y} \rangle = 0.$$

Ex: For Euclidean inner product

$$\vec{x}^T \vec{y} = 0$$

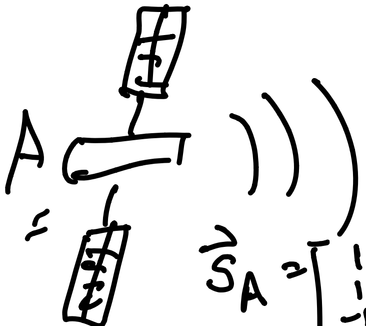
$$= \|\vec{x}\| \, \|\vec{y}\| \cos\theta$$

euclidean norms $\quad$ angle btwn $\vec{x}, \vec{y}$

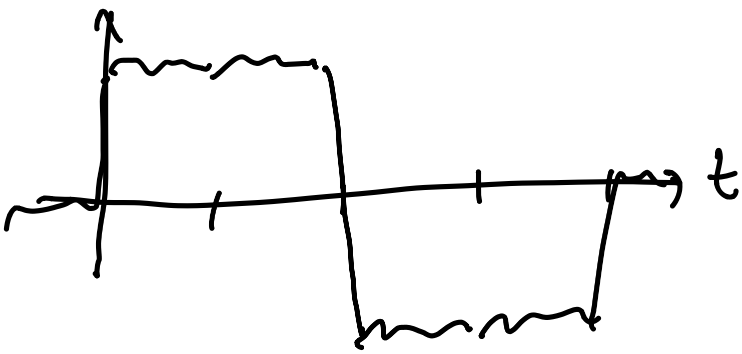
$\Rightarrow$ $\vec{x}, \vec{y}$ are perpendicular (i.e. $\theta = \frac{\pi}{2}$)

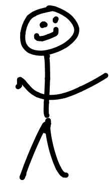
Application to Classification

Problem: classify received signal to determine which satellite is transmitting.



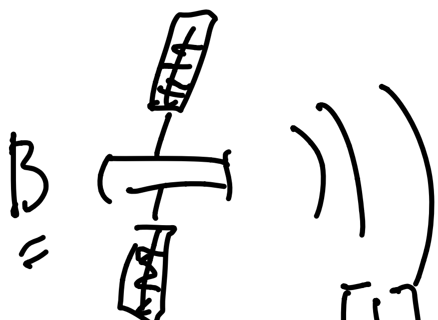
$$\vec{s}_A = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$



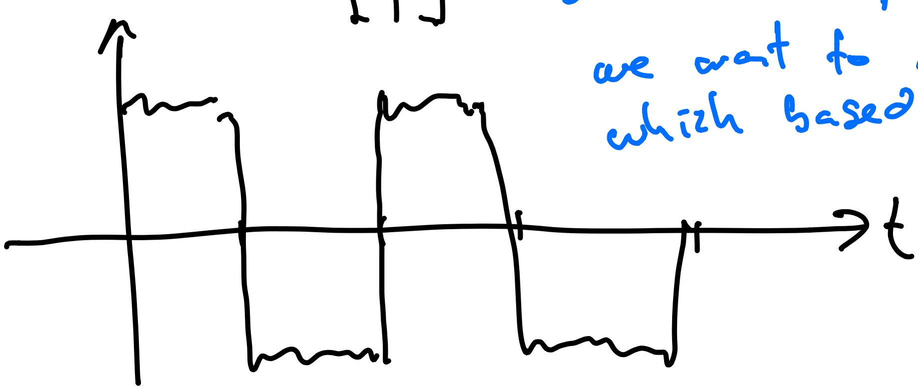


rcvd signal

$$\vec{r} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}}_{\text{satellite signature}} + \underbrace{\begin{bmatrix} .1 \\ -.2 \\ -.3 \\ .2 \end{bmatrix}}_{\text{noise}}$$



$$\vec{s}_B = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$



Q: If I don't know what satellite is transmitting, How do I figure it out?

I observe $\vec{r} = \vec{s}_i + \vec{n}$ where $i \in \{A, B\}$, and we want to determine which based on $\vec{r}$.

observation

Q: How to mathematically formulate this classification problem?

One method: minimize the "error" vector $\|\cdot\|$.

Mathematically:

$$i^* = \arg\min_{i \in \{A,B\}} \overbrace{\|\vec{r} - \vec{s}_i\|^2}^{\langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle}$$

index I declare as which satellite was transmitting.

received signal

signature of satellite i.

In words: choose satellite whose signature is closest to what I received.

$$\underbrace{\langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle}_{\text{minimize}} = \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$

$$= \langle \vec{r}, \vec{r} \rangle - \langle \vec{r}, \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} \rangle + \langle \vec{s}_i, \vec{s}_i \rangle$$

$$= \underbrace{\|\vec{r}\|^2}_{\text{fixed}} + \underbrace{\|\vec{s}_i\|^2}_{\text{equal for each } i \text{ by design (under assumption of Euclidean norm)}} - 2\underbrace{\langle \vec{r}, \vec{s}_i \rangle}_{\text{maximize}}$$

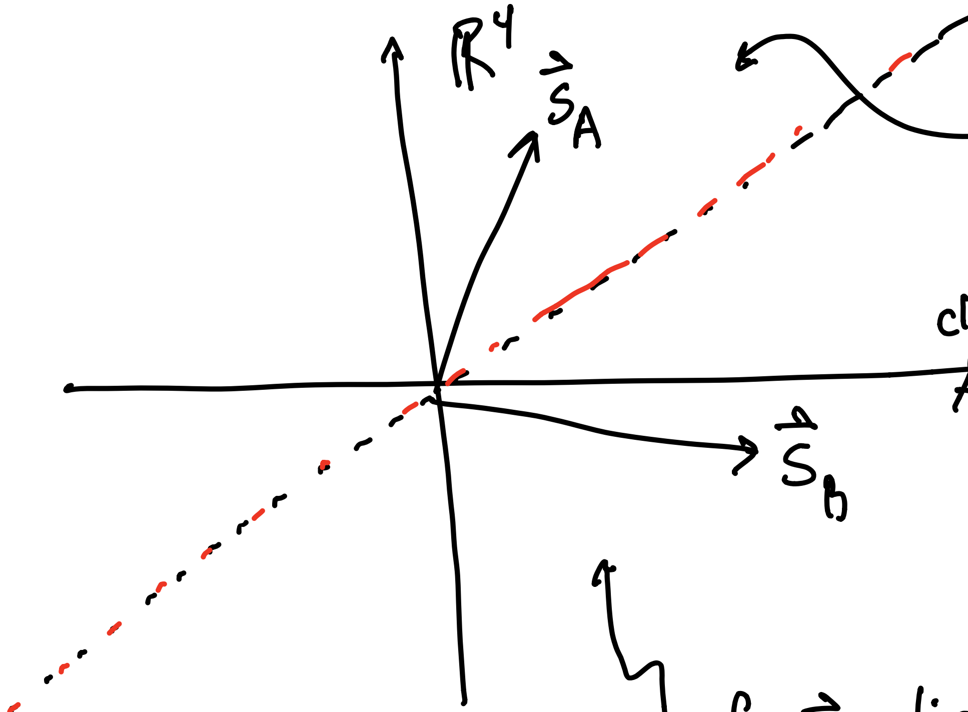
So, minimizing $\|\vec{r} - \vec{s}_i\|$ over $i \in \{A, B\}$ is equivalent to maximizing $\langle \vec{r}, \vec{s}_i \rangle$ over $i \in \{A, B\}$.

Classification Procedure

For $i \in \{A, B\}$
    compute $\langle \vec{r}, \vec{s}_i \rangle$

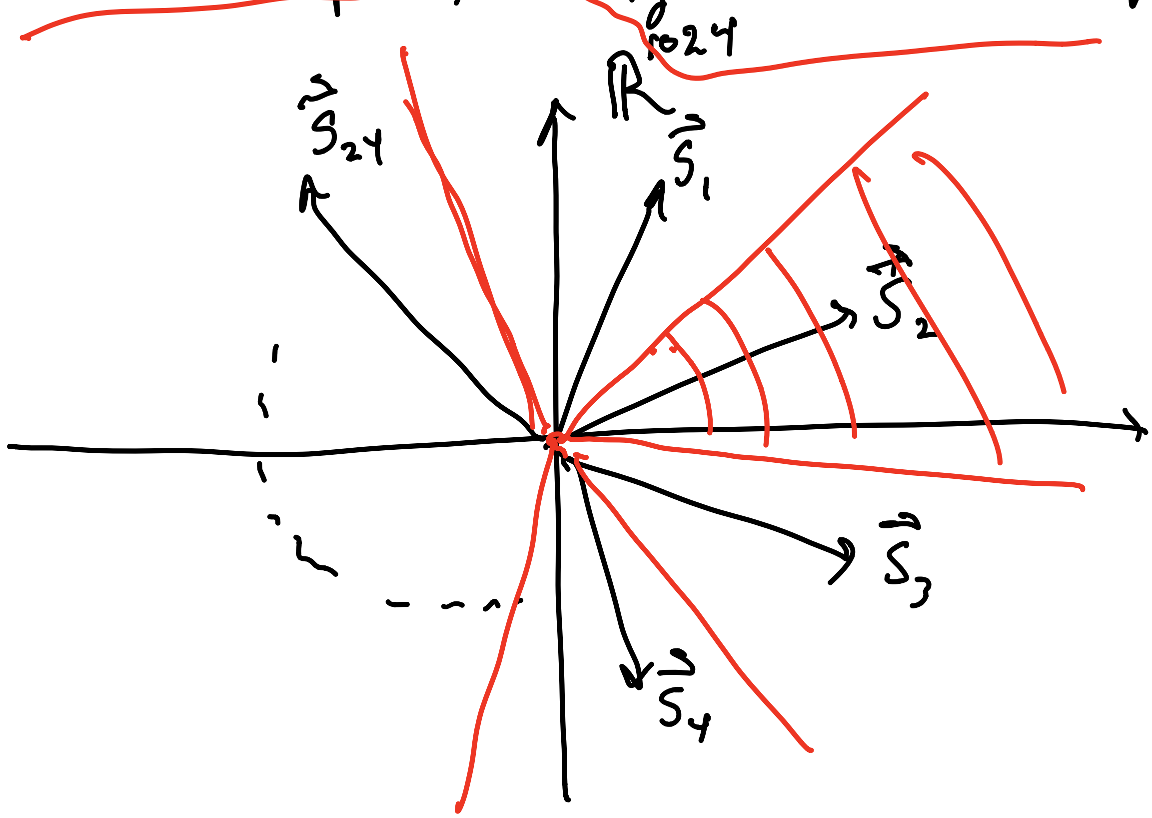
Return index i that maximizes this.

Summarize: Inner products quantitatively capture notion of vector "similarity", which can be exploited in applications.



if $\vec{r}$ lies in this halfspace, classify as A transmitting

if $\vec{r}$ lies in this halfspace, classify as B transmitting



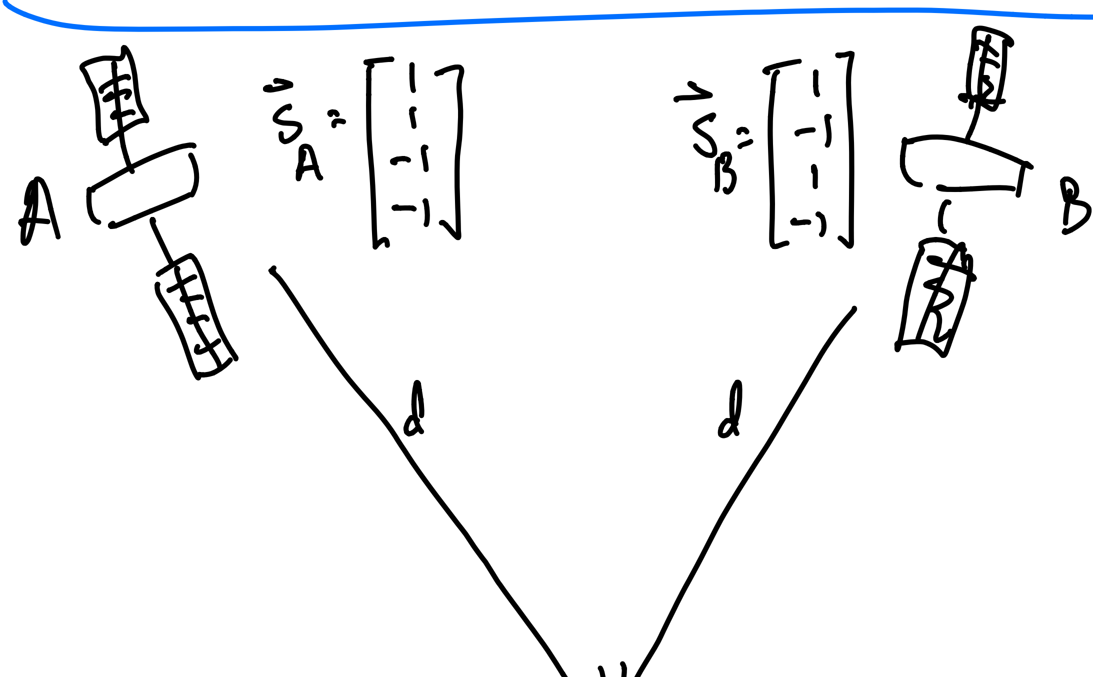
For $i = 1, \cdots, 24$

compute $\langle \vec{r}, \vec{s}_i \rangle$

Return maximizing index i.

Two questions: 1) Interference.

2) Timing.



Possibility 1: Both satellites transmitting

Possibility 2: Only A transmitting

Possibility 3: Only B transmitting.

Basic Procedure: Compute $\langle \vec{r}, \vec{s}_i \rangle$ to determine whether $\vec{s}_i$ is present.

(i.e. satellite i is transmitting)

Assume Henceforth $\langle \cdot, \cdot \rangle =$ Euclidean inner product.

Situation 1: $\vec{r} = \vec{s}_A + \vec{s}_B + \vec{n}$.

$$\langle \vec{r}, \vec{s}_A \rangle = \langle \vec{s}_A + \vec{s}_B + \vec{n}, \vec{s}_A \rangle$$

$$= \langle \vec{s}_A, \vec{s}_A \rangle + \langle \vec{s}_B, \vec{s}_A \rangle + \langle \vec{n}, \vec{s}_A \rangle$$

$$= 4 + \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \text{"small"}.$$

$$\approx 4$$

$$\langle \vec{r}, \vec{s}_B \rangle = \langle \vec{s}_A + \vec{s}_B + \vec{n}, \vec{s}_B \rangle$$

$$\approx 4$$

Situation 2: $\vec{r} = \vec{s}_A + \vec{n}$.

$$\langle \vec{r}, \vec{s}_A \rangle \approx 4$$

$$\langle \vec{r}, \vec{s}_B \rangle = \langle \vec{s}_A + \vec{n}, \vec{s}_B \rangle$$

$$= \underbrace{\langle \vec{s}_A, \vec{s}_B \rangle}_{0} + \underbrace{\langle \vec{n}, \vec{s}_B \rangle}_{\text{"small"}}$$

$$\approx 0$$

Situation 4: $\vec{r} = \vec{n}$ (no satellite transmitting)

$$\langle \vec{r}, \vec{s}_A \rangle \approx 0$$

$$\langle \vec{r}, \vec{s}_B \rangle \approx 0$$

Big Idea: Design signatures to be orthogonal. This allows us to "separate out" interfering signals.

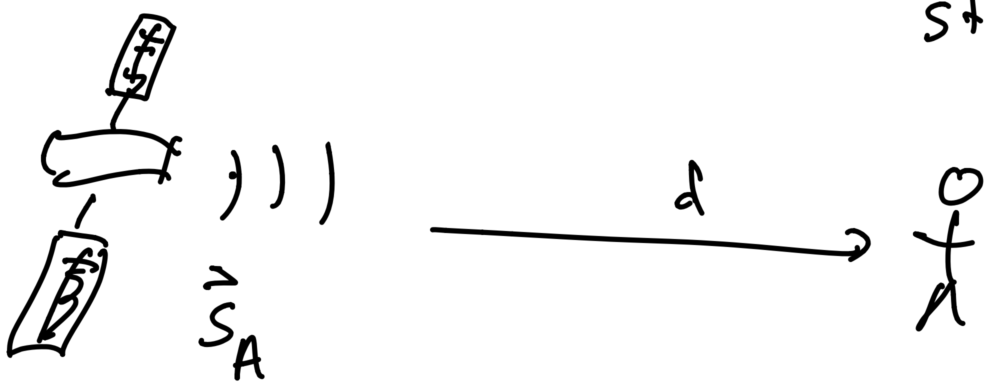
In cell systems you've probably heard the term "OFDM". "CDMA"

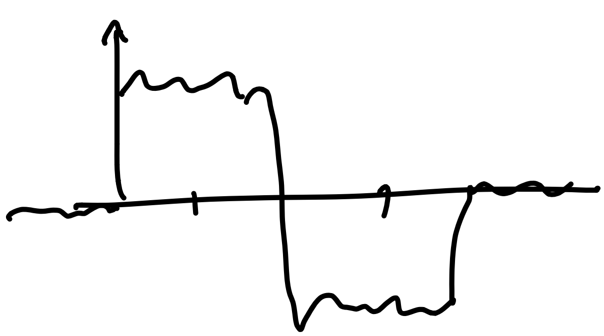
orthogonal frequency division multiplexing

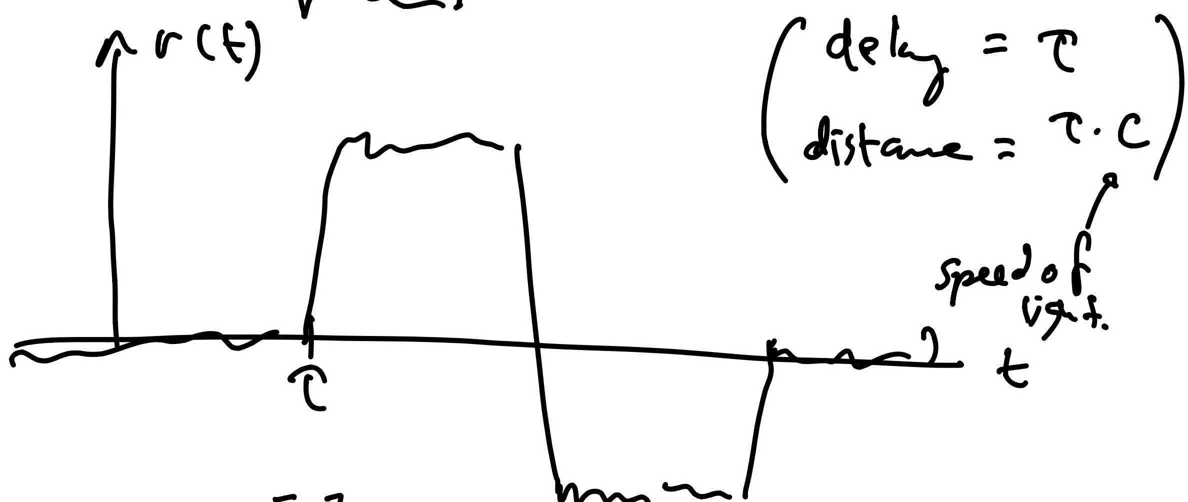
Second Big Question:

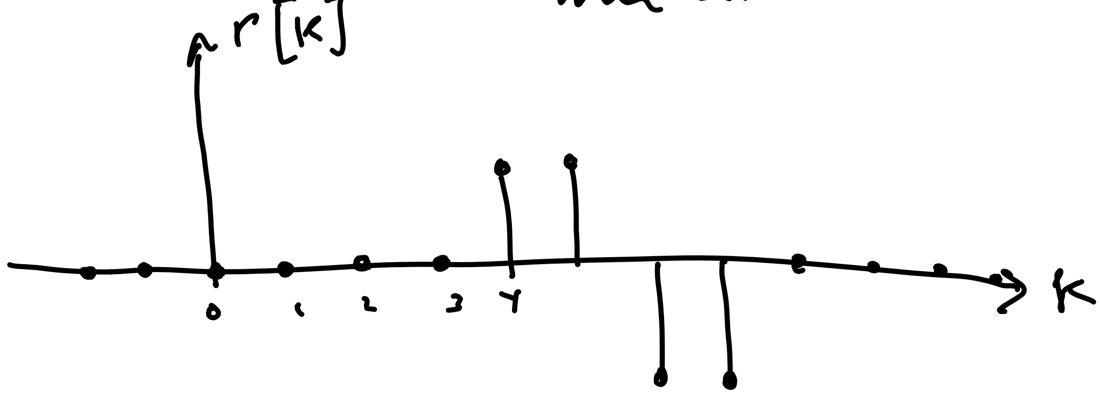
What about timing?

i.e. how do I know when transmission started?









To determine delay systematically,

compute $\left\langle [r[i], \ldots, r[i+3]],\ \vec{S}_A \right\rangle$

for each $i = -\infty$ to ∞.

Def: For sequences $x[n]$, $y[n]$

$n \in$ Integers

Define cross-correlation as:

$$\underbrace{\mathrm{corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]\, y[i-k].}$$

looks scary, but idea is just

$\mathrm{corr}_x(y)[k]$ is inner product between x and y "shifted by k".

Examples to come next time!