Today: Wrap-up of Matching Pursuit / Module 3!

Objective: Solve (exactly or approximately) the problem:

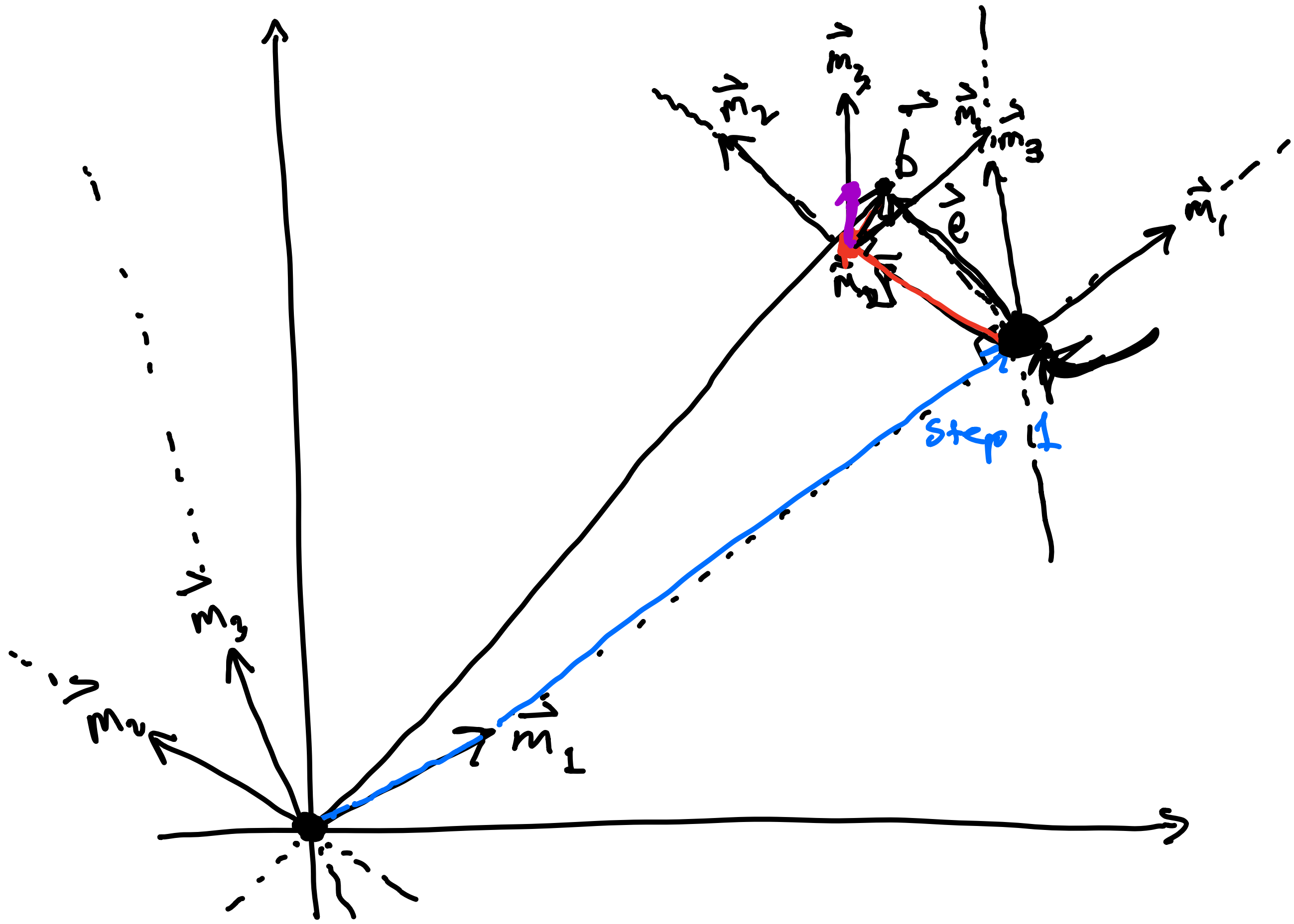
$$\min_{\vec{x}} \|M\vec{x} - \vec{b}\| \quad \text{subject to} \quad \underbrace{\|\vec{x}\|_0 \leq k}_{\text{Sparsity constraint}}.$$

"# of nonzero entries in $\vec{x}$ $\leq k$"

M = "dictionary", we assume $\|\vec{m}_i\| = 1$.

$$\underbrace{\begin{bmatrix} | & & | \\ \vec{m}_1 & \cdots & \vec{m}_m \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}}_{\sum_{i=1}^{m} \vec{m}_i x_i} \approx \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$\sum_{i=1}^{m} \vec{m}_i x_i \approx \vec{b}$$



<u>Initialize</u>: $\vec{e} = \vec{b}$, $\vec{x} = 0$

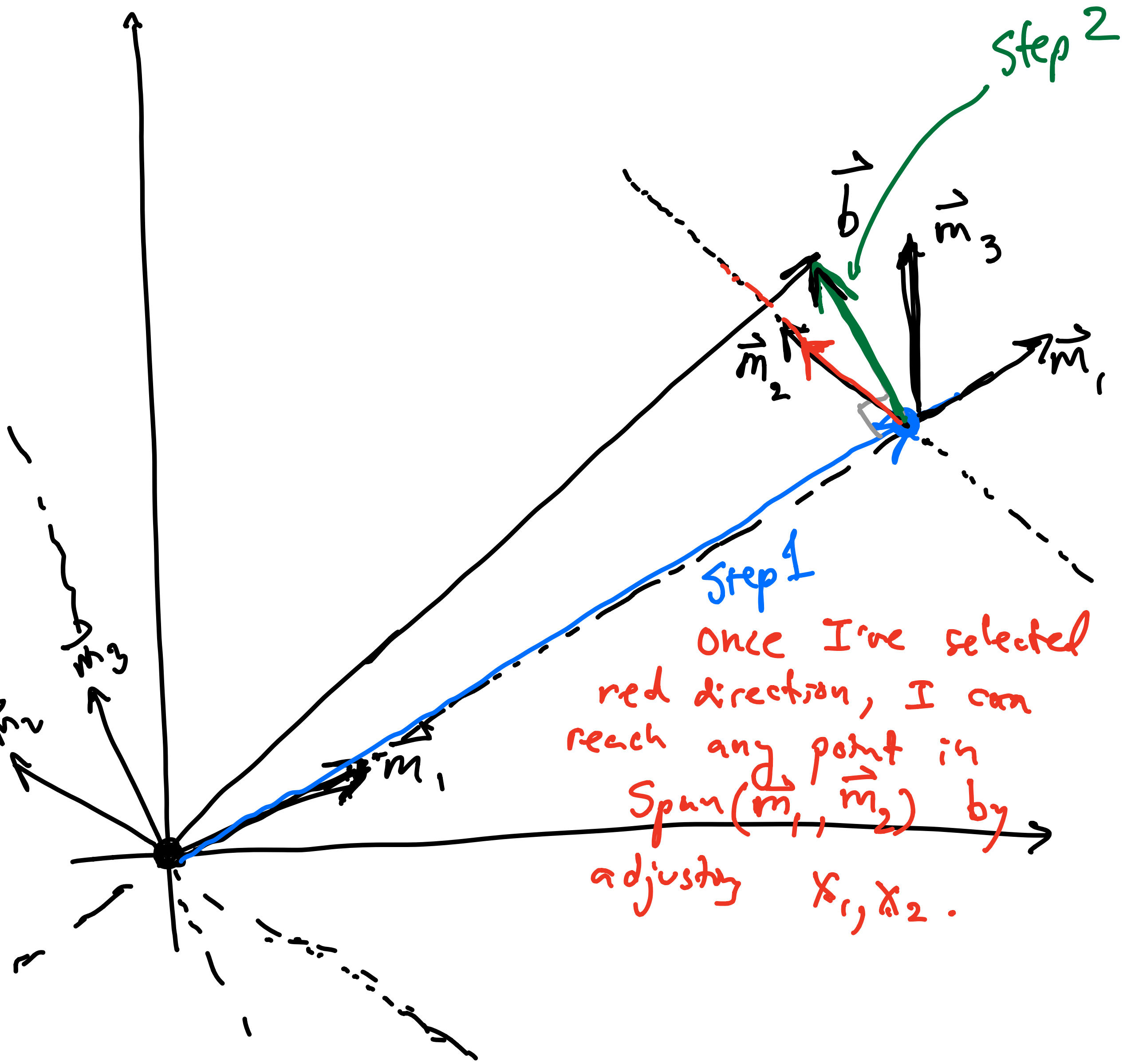
For $j = 1 \cdots k$

- Find i such that $|\langle \vec{m}_i, \vec{e} \rangle|$ is maximum
  (i.e. find allowed direction of travel that gets me closest to destination)
- Update $x_i = x_i + \langle \vec{e}, \vec{m}_i \rangle$
  (i.e. move in that direction the optimal amount)

• Update $\vec{e} = \vec{b} - M\vec{x}$ (= residual error)

End.

Orthogonal Matching Pursuit (OMP) is a variation of MP.



Once I've selected red direction, I can reach any point in $\mathrm{Span}(\vec{m}_1, \vec{m}_2)$ by adjusting x_1, x_2.

OMP:

Initialize: $\vec{e} = \vec{b}$. $A = [\,]$.

For $j = 1 \dots k$

- Find index i that maximizes $|\langle \vec{m}_i, \vec{e} \rangle|$

  (choose a direction $\vec{m}_i$ such that moving in that direction would get me closest to $\vec{b}$.

  $\iff \min_i \| \vec{e} - \mathrm{proj}_{\vec{m}_i}(\vec{e}) \|$)

- Update $A = [A \mid \vec{m}_i]$

  (cols of A consist of directions we are willing to travel in thus far)

- Update $\vec{e} = \vec{b} - \underbrace{A(A^T A)^{-1} A^T \vec{b}}$.

  (vector in $\mathrm{col}(A) = \mathrm{span}$(cols $\vec{m}_i$ selected so far) that is closest to $\vec{b}$.

i.e. $(A^TA)^{-1}A^T\vec{b} = \underset{\vec{v}}{\arg\min} \underbrace{\|A\vec{v} - \vec{b}\|}$.

End

At conclusion of procedure, we have

$$A = \left[\vec{m}_{i_1}, \vec{m}_{i_2}, \cdots \vec{m}_{i_k}\right]$$

indices of cols selected are

$$F = \{i_1, i_2, \cdots i_k\}.$$

Output $\vec{x}$ by defining nonzero entries

$$\begin{bmatrix} x_{i_1} \\ \vdots \\ x_{i_k} \end{bmatrix} = (A^TA)^{-1}A^T\vec{b}$$

LS solution to

$$\min_{\vec{x}} \|M\vec{x} - \vec{b}\| \quad \text{s.t.} \quad x_i = 0 \text{ unless } i \in F.$$

Q: How do we know that $(A^TA)^{-1}$ exists?

$\Leftrightarrow$ requiring cols of A to be Linearly Independent.

<u>A</u>: A will <u>always</u> have linearly indep. cols. (Why?)

orthogonal to

In particular $\vec{e} \perp col(A)$ by orthogonality property of LS solution.

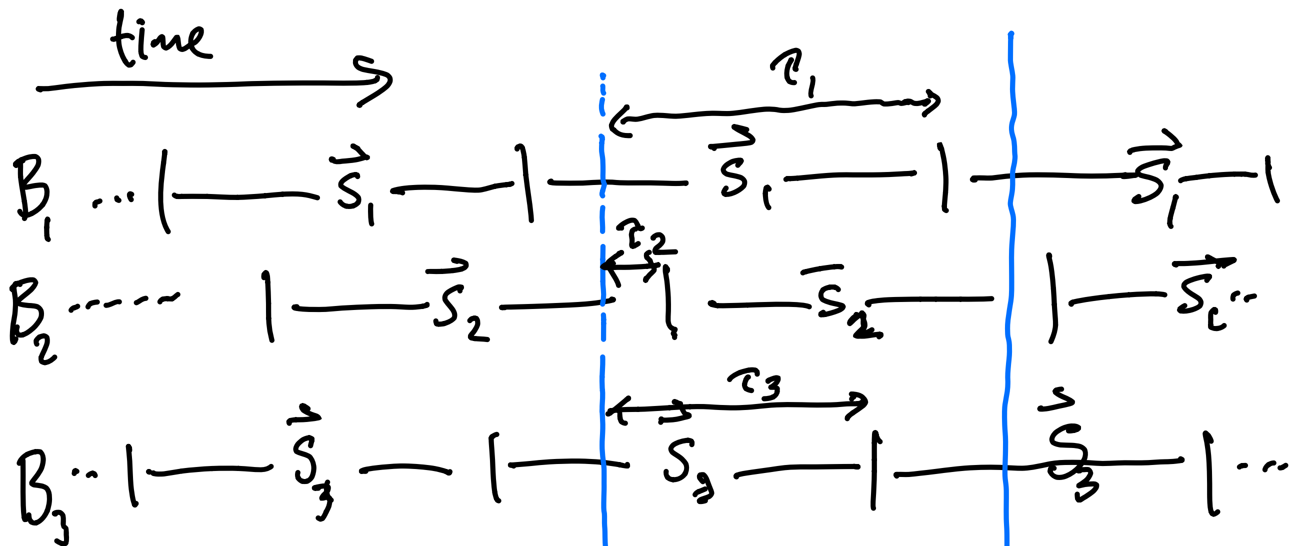
So, if $\vec{m}_j \in col(A)$, then would imply $\langle \vec{e}, \vec{m}_j \rangle = 0$, so we would select some other column*.

(* if $\langle \vec{e}, \vec{m}_j \rangle = 0 \ \forall j$, then we would just quit because we are already at LS solution)

<u>Application to localization:</u>

Say $\vec{s}_1, \vec{s}_2, \vec{s}_3$ are signatures of length N, normalized so that $\|\vec{s}_1\| = \|\vec{s}_2\| = \|\vec{s}_3\| = 1$

Beacons transmit these signatures on a loop.



Receiver records a sequence of N observations

$$\vec{r} = \vec{s}_1^{(\tau_1)} + \vec{s}_2^{(\tau_2)} + \vec{s}_3^{(\tau_3)} + \vec{n}$$

$\vec{n}$ = noise

For vector $\vec{x}$, I write $\vec{x}^{(\tau)}$ to denote cyclic shift of $\vec{x}$ by τ.

Ex:

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \vec{x}^{(0)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \vec{x}^{(1)} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{x}^{(2)} = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \quad \vec{x}^{(3)} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

We can set up the recovery problem using MP (or OMP).

$$M = \begin{bmatrix} | & | & & | & | & & | & | & & | \\ \vec{s}_1^{(0)} & \vec{s}_1^{(1)} & \cdots & \vec{s}_1^{(N-1)}, & \vec{s}_2^{(0)} & \cdots & \vec{s}_2^{(N-1)} & \vec{s}_3^{(0)} & \cdots & \vec{s}_3^{(N-1)} \\ | & | & & | & | & & | & | & & | \end{bmatrix}$$

$$M \in \mathbb{R}^{N \times 3N}$$

Now, want to solve

$$\min_{\vec{x}} \underbrace{\| M\vec{x} - \vec{r} \|} \quad \text{s.t.} \quad \|\vec{x}\|_0 \leq 3.$$

$$\downarrow$$

$$\left\| M\vec{x} - \left(\vec{s}_1^{\,(\tau_1)} + \cdots + \vec{s}_3^{\,(\tau_3)} + \vec{n}\right) \right\|$$

with any luck will find solution

$$\vec{x} = \begin{array}{c} \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array}\right] \end{array} \begin{array}{l} \\ \\ \\ \leftarrow \text{position } \tau_1 \\ \\ \\ \\ \\ \\ \\ \leftarrow \text{position } N + \tau_2 \\ \\ \\ \\ \\ \\ \\ \leftarrow \text{position } 2N + \tau_3 \\ \\ \\ \\ \end{array}$$

$$M\vec{x} = \vec{s}_1^{\,(\tau_1)} + \cdots + \vec{s}_3^{\,(\tau_3)}$$

Note: Generalizes to k beacons, if some aren't present, this will be reflected in solution $\vec{x}$ (e.g. some values close to 1, others very small)

<u>Remark</u>: Gold codes have good correlation properties by design. This helps MP/OMP find the "right" solution.