EEC S 16A
2/13/2020

- Nullspace & Column space (Subspaces)
- Determinants.

①

Recall:

Column Space of a matrix $A \in \mathbb{R}^{m \times n}$

$$C(A) = \text{span}\{\underbrace{\vec{a}_1 \cdots \vec{a}_n}_{\text{columns of } A}\} = \text{range}(A)$$

$$= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

$C(A)$ is a subspace of $\mathbb{R}^m$

Nullspace: Given $A \in \mathbb{R}^{m \times n}$, nullspace of A is defined as

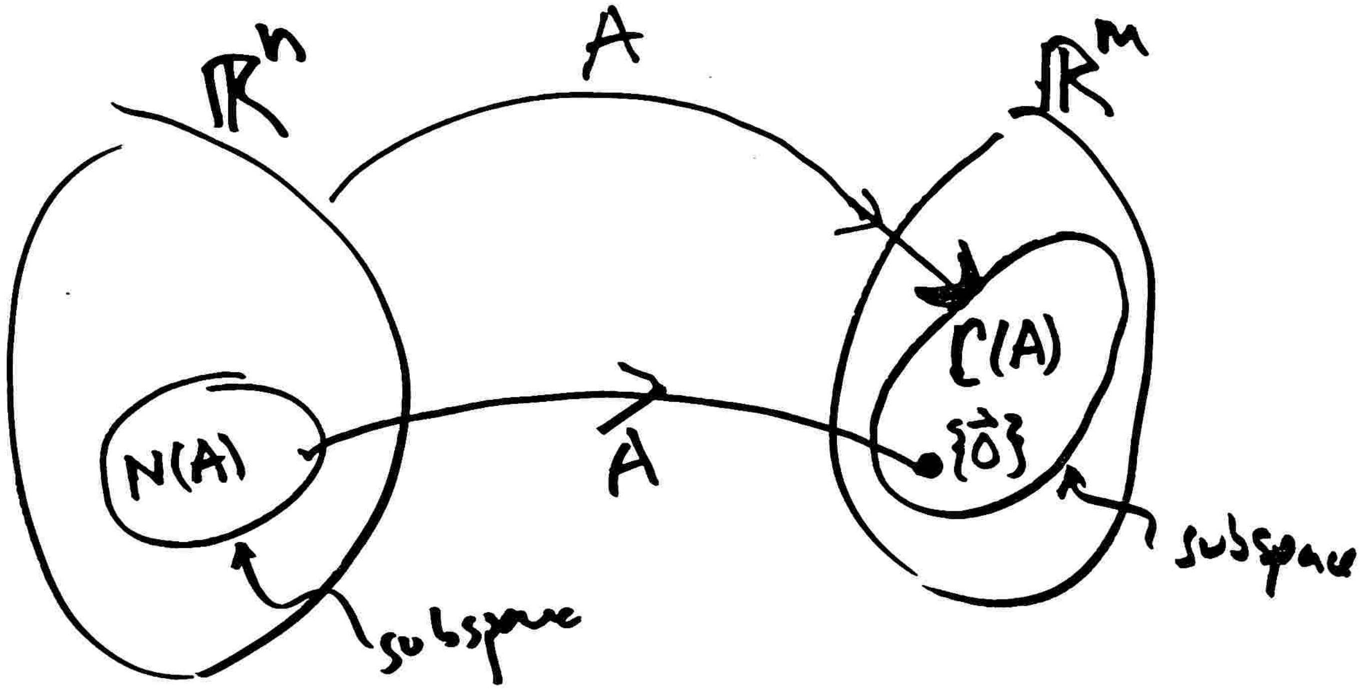
$$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0 \} \subset \mathbb{R}^n$$

Claim: $N(A)$ is a subspace of $\mathbb{R}^n$.

Pf: Let $\vec{x}, \vec{y} \in N(A)$, for any scalars $\alpha, \beta \in \mathbb{R}$

$$A(\alpha\vec{x} + \beta\vec{y}) = \alpha A\vec{x} + \beta A\vec{y} = 0$$

$$\Rightarrow \alpha\vec{x} + \beta\vec{y} \in N(A) \qquad \blacksquare$$



$\mathbb{R}^n$
A
$\mathbb{R}^m$
$C(A)$
$N(A)$
A
$\{\vec{0}\}$
subspace
subspace

Q: How to compute a subspace?

A: Find a basis!

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{bmatrix}$$

$$C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \end{bmatrix} \right\} = \mathbb{R}^2$$

$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$N(A) = \{ \vec{x} : A\vec{x} = \vec{0} \}$$

= all solutions to sys. of LEQs $A\vec{x}=\vec{0}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 2 & 4 & 6 & 8 & 0 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 0 & -2 & -4 & -6 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{2}R_2} \left[\begin{array}{cccc|c} 1 & 3 & 5 & 7 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

$$N(A) = \left\{ \vec{x} : \begin{array}{l} x_1 \quad - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{array} \right\}$$

$$\begin{aligned} x_1 &= x_3 + 2x_4 \\ x_2 &= -2x_3 - 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} x_4 .$$

every vector in nullspace = linear comb. of these 2 vectors

i.e., a basis for $N(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

$\dim(C(A)) = 2$ $\quad\quad$ $\dim(N(A)) = 2$

$$\dim(C(A)) + \dim(N(A)) = n$$

Remark: dimension of $N(A)$

$= \#$ free variables after GE for $A\vec{x}=0$

$= \#$ of "degrees of freedom" in solution to $A\vec{y}=\vec{0}$.

Example: $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{bmatrix}$ $\quad C(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\dim(C(A)) = 2.$

$N(A) = \{ \vec{x} : A\vec{x} = 0 \}$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & -3 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$
$x_2 = 0$

$N(A) = \{\vec{0}\}$.

$\dim(N(A)) = 0$

$$\underbrace{\dim(N(A))}_{0} + \underbrace{\dim(C(A))}_{2} = \underbrace{n}_{2}.$$

$$A = \begin{bmatrix} 1 & 4 & 5 & 9 \\ 2 & 4 & 6 & 10 \\ 3 & 4 & 7 & 11 \\ 4 & 4 & 8 & 12 \end{bmatrix} \xrightarrow{A\vec{x}=\vec{0}} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

cols for basic variables

$$\text{basis for } C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \right\}$$

$$N(A) = \left\{ \begin{array}{l} x_1 = -x_3 - x_4 \\ x_2 = -x_3 - 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \right\} = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Once again: $\underbrace{\dim(C(A))}_{2} + \underbrace{\dim(N(A))}_{2} = \underbrace{n}_{4}$.

The pattern we observed holds in general.

<u>The rank-nullity theorem</u>: Let V, W be (finite-dimensional) vector spaces, and let A be a linear transformation $A: V \to W$.

$$\dim(N(A)) + \dim(\text{range}(A)) = \dim(V)$$

"rank" $= \dim(\text{range}(A)) = \dim(C(A))$

"nullity" $= \dim(N(A))$.

1) $N(A)$ helps characterize *all* solutions to a system of LEQs $A\vec{x}=\vec{b}$.

2) $C(A)$ characterizes consistency of a system of LEQs $A\vec{x}=\vec{b}$.

[consistent iff $\vec{b}\in C(A)$] (obvious)

Suppose $A\vec{x}=\vec{b}$ is a consistent system of LEQs, and you are given a particular soln. $\vec{x}_0$: $A\vec{x}_0=\vec{b}$

Then $\{$all solutions$\} = \{\vec{x}_0+\vec{v} \mid \vec{v}\in N(A)\}$

Pf: Take any solution $\vec{x}_1$: $A\vec{x}_1 = \vec{b}$

WTS: $\vec{x}_1 = \vec{x}_0 + \vec{v}$ for some $\vec{v} \in N(A)$.

Note: $\vec{x}_1 = \vec{x}_0 + (\vec{x}_1 - \vec{x}_0)$, done if we show $\vec{x}_1 - \vec{x}_0 \in N(A)$

$$A(\vec{x}_1 - \vec{x}_0) = A\vec{x}_1 - A\vec{x}_0 = \vec{b} - \vec{b} = \vec{0}.$$

$$\Rightarrow (\vec{x}_1 - \vec{x}_0) \in N(A) \quad \checkmark$$

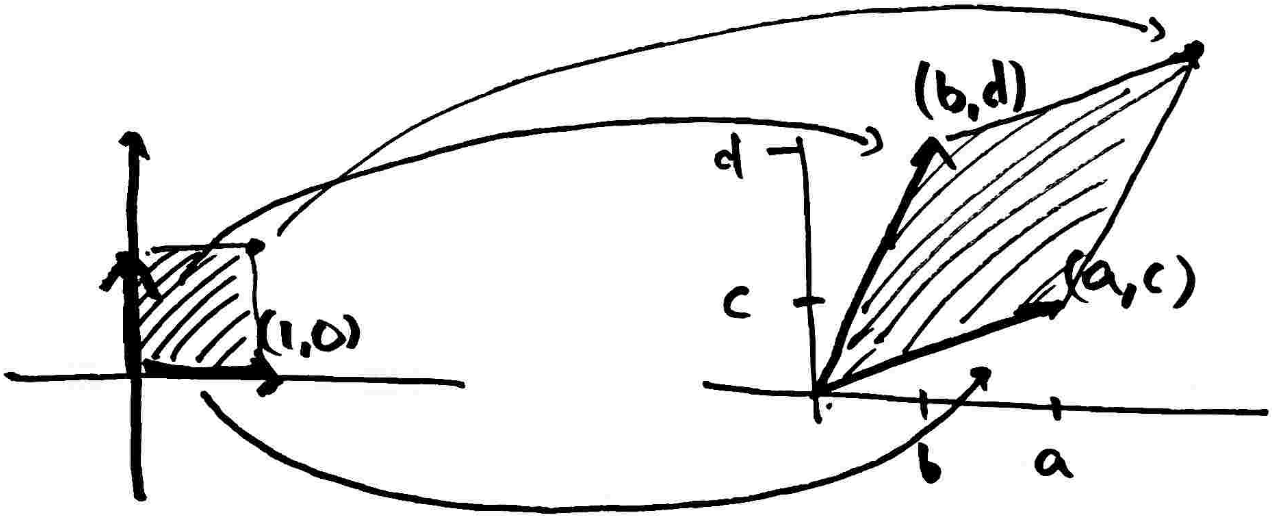
Let $A \in \mathbb{R}^{n \times n}$. A scalar λ and a vector $\vec{x}$ are called an eigenvalue (resp. eigenvector) for A if

$$A\vec{x} = \lambda\vec{x}.$$

Determinant is a function of a square matrix, denoted $\det(A)$ or $|A|$.

For 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$

determinant describes "change of volume".



area = 1 $\qquad\qquad$ area = ad − bc

<u>Rmk</u>: det(A) can be negative,
so det(A) corresponds to "signed" volume

<u>What we care about</u>: $\underbrace{\det(A) \neq 0}_{A \text{ invertible}}$ or $\underbrace{= 0}_{A \text{ not invertible}}$