EECS 16A
2/18/2020

- Determinants
- Eigenvalues/vectors

MT1
office Hrs

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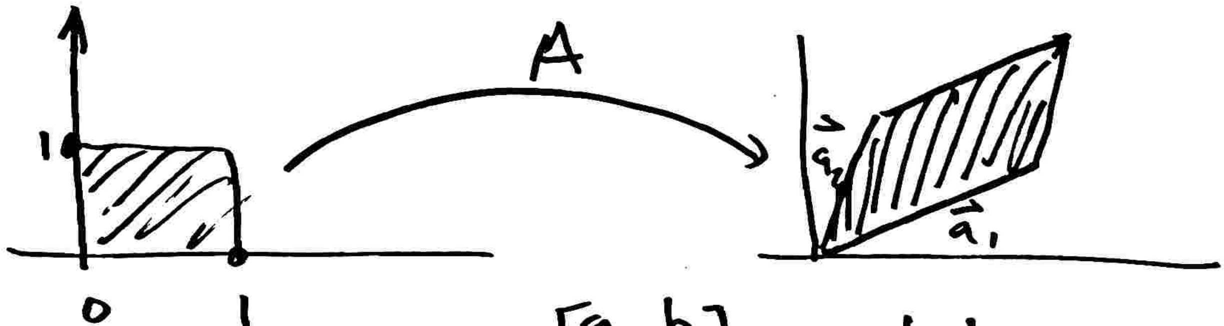
Last time: For $A \in \mathbb{R}^{n \times n}$, a pair $(\lambda, \vec{x})$ is called an eigenvalue-eigenvector pair if it satisfies

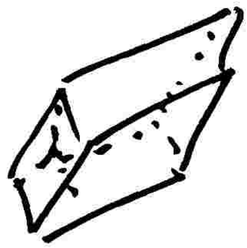
$$A\vec{x} = \lambda \vec{x}.$$

Determinant of matrix is (signed) volume of unit cube's image under A:



For 2x2 matrix A: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$



<u>Important Observation</u>: $\det(A)$ is a polynomial of degree n in the entries of A.

$\det(A) = 0 \iff A$ is <u>not invertible</u>
singular

$\det(A) \neq 0 \iff A$ is invertible

Q: Why should I care about determinants?

A: They help us identify eigenvalues of A.

Q: Why should I care about eigenvalues? or eigenvectors?

Motivating Application: Page Rank.

$$\vec{x}(0) = \begin{bmatrix} \text{frac of users @ } S \\ \text{frac of users @ } B \end{bmatrix}$$

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \vec{x}(1) = Q\vec{x}(0) \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$\vec{x}(2) = Q\vec{x}(1) = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}, \quad \vec{x}(3) = \begin{bmatrix} 1/8 \\ 7/8 \end{bmatrix} \quad \cdots \quad \vec{x}(t) = \begin{bmatrix} (1/2)^t \\ 1-(1/2)^t \end{bmatrix}$$

$t \to \infty \quad \vec{x}(\infty) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

This is called "steady state" distribution of users
or "equilibrium"
or "invariant vector".

$$\vec{x}_{steady} = Q\vec{x}_{steady}$$

Note: $\vec{x}_{steady}$ is an eigenvector of Q with eigenvalue 1.

Q: How can we compute the set of possible invariant vectors / equilibriums / steady states for a given matrix Q?

A: $Q\vec{x}_{steady} - I\vec{x}_{steady} = 0$

$(Q-I)\vec{x}_{steady} = 0$

$\Rightarrow$ compute nullspace of $(Q-I)$

$$\left[Q-I \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix}-1/2 & 0\\ 1/2 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\0\end{matrix}\right] \rightarrow \left[\begin{matrix}1 & 0\\ 1/2 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\0\end{matrix}\right] \rightarrow \left[\begin{matrix}1 & 0\\ 0 & 0\end{matrix}\,\middle|\,\begin{matrix}0\\0\end{matrix}\right]$$

$x_1 = 0$

$x_2 = \text{free}$

$$N(Q-I) = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$$

= "eigenspace of Q, corresponding to eigenvalue 1".

Reason for terminology "eigenspace"?

The set of vectors $\vec{x}$ that are eigenvectors corresponding to eigenvalue λ form a subspace.

Pf: $A\vec{x} = \lambda\vec{x}$ $\quad A\vec{y} = \lambda\vec{y}$, $\vec{x}, \vec{y}$ nonzero vectors.

$$A(\alpha\vec{x} + \beta\vec{y}) = \alpha A\vec{x} + \beta A\vec{y} = \alpha\lambda\vec{x} + \beta\lambda\vec{y}$$
$$= \lambda(\alpha\vec{x} + \beta\vec{y}) \quad \checkmark$$

$\Rightarrow$ $E_\lambda = \{\vec{x} : A\vec{x} = \lambda\vec{x}\}$ is a subspace,

called eigenspace of A corresp. to eigenvalue λ.

Q: How do we compute Eigenspaces in general?

$$E_\lambda = \underbrace{N(Q-\lambda I)}$$

$$\vec{x} \in N(Q-\lambda I) \iff Q\vec{x} = \lambda\vec{x}.$$

Note: we need to be able to determine eigenvalues of Q in order to solve above

Q: How do we determine eigenvalues of a matrix A?

we need $(A-\lambda I)$ to have nontrivial nullspace

$\iff (A-\lambda I)$ is not invertible.

$\iff$ need $\det(A-\lambda I) = 0$.

So: <u>Idea</u> = use determinant of $A - \lambda I$ to solve for eigenvalues.

$$\underbrace{P_A}_{\text{characteristic polynomial of } A}(\lambda) = \det(A - \lambda I) = \text{polynomial of degree } n \text{ in } \lambda.$$

λ eigenvalue of A $\Leftrightarrow$ $\det(A - \lambda I) = 0$ $\Leftrightarrow$ λ is root of P_A

<u>Fundamental Thm of Algebra</u>: polynomial of degree n has exactly n roots (counting multiplicity)

$$P_A(\lambda) = \prod_{i=1}^{n} (\lambda - \underbrace{\lambda_i}_{\text{roots.}})$$

<u>Note</u>: eigenvalues can be complex!

The n roots of p_A are the eigenvalues of A.

Lets go back to $Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$

Q: what are eigenvalues of Q?

$$\det(Q - \lambda I) = \det\begin{pmatrix} 1/2-\lambda & 0 \\ 1/2 & 1-\lambda \end{pmatrix} = \left(\tfrac{1}{2}-\lambda\right)(1-\lambda)$$

$$\shortparallel$$

$$0 \quad \lambda_1 = 1,\ \lambda_2 = 1/2,$$

previously, we saw $E_1 = \text{span}\Big\{\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}_1}\Big\}$.

Q: what is eigenspace $E_{1/2} = ?$

$$E_{\frac{1}{2}} = N\left(\begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}\right) = \text{span}\Big\{\underbrace{\begin{bmatrix} +1 \\ -1 \end{bmatrix}}_{\vec{v}_2}\Big\}$$

observe:

$\vec{v}_1, \vec{v}_2$ are linearly independent.

$$\vec{x}(0) = \begin{bmatrix} \theta \\ 1-\theta \end{bmatrix} \qquad \theta \in [0,1]$$

$$= 1\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{v}_1} + \theta\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{v}_2} = \text{"eigen decomposition"}$$

$$\vec{x}(t) = Q^t \vec{x}(0) = 1 Q^t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \theta Q^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \theta \left(\tfrac{1}{2}\right)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \theta (1/2)^t \\ 1 - \theta (1/2)^t \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ compute eigenvalues / eigenspaces of A.

$$\det \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5$$

$$= (\lambda - 5)(\lambda + 1) = 0$$

$$\lambda_1 = 5, \quad \lambda_2 = -1$$

$$E_5 = N\left(\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\right)$$

$$\left[\begin{array}{cc|c} -4 & 2 & 0 \\ 4 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$x_1 = + x_2/2$$

$$E_5 = \text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}}_{A} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 + 2 \\ 2 + 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

$$E_{-1} = N\left(\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}\right) = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

check $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.