MT1 : Mon 3/1 7-9 PM

Review Session: Thurs 2/25 7-9 PM

We are starting Module 2! Please read Note 11.

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"Eigenstuffs" $A\vec{x} = \lambda\vec{x} \rightsquigarrow (A-\lambda I)\vec{x} = \vec{0}$

① Eigenvalues $\lambda_i \rightarrow$ solve for $\lambda_i$ st $\det(A-\lambda I) = 0$

② Eigenvectors $\vec{x}_i \rightarrow$ solve for nullspace of $A - \lambda_i I$

Be comfortable
(1) Mechanical solve
(2) Transition Matrices "Pumps"
(3) Steady State

Change of Basis (COB)

Basis in $\mathbb{R}^N \sim$ ~~(1)~~ set of $\underline{\underline{N}}$ $\underline{\underline{\text{L.I.}}}$ vectors

elem. basis / std basis. $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

$\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

$$\vec{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \underbrace{r_1\vec{e}_1 + r_2\vec{e}_2 + \dots + r_n\vec{e}_n}_{\text{L.C. of basis vectors}} = \begin{bmatrix} | & & | \\ \vec{e}_1 & \dots & \vec{e}_n \\ | & & | \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \mathcal{U}\vec{r}$$

$V \sim \{\vec{v}_1, \vec{v}_2 \dots, \vec{v}_n\}$

$$\vec{r} = \underbrace{r_1^{(v)}\vec{v}_1 + r_2^{(v)}\vec{v}_2 + \dots + r_n^{(v)}\vec{v}_n}_{\text{LC of V basis vectors}} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} r_1^{(v)} \\ \vdots \\ r_n^{(v)} \end{bmatrix} = V\vec{r}^{(v)}$$

$$\mathcal{U}\vec{r} = V\vec{r}^{(v)} \Rightarrow \vec{r}^{(v)} = \underbrace{V^{-1}\mathcal{U}}_{\text{COB}}\vec{r}$$

into the V basis.

out of the $\mathcal{U}$ basis

Transform in other bases.

some transform

$$A = I A I^{-1}$$

into I basis

out of I basis, back to original basis

$$A = V A^{(v)} V^{-1}$$

into V basis

original transform

the transform in the V basis

out of V basis, back to original

Compression

Consider $A = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 0.01 \end{bmatrix} \approx \begin{bmatrix} 10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda = 10, 15, 0.01$ $\qquad$ $\lambda = \underbrace{10, 15}, \underline{0}$

If we can look at just the eigenvalues of our transform, we can select "significant" information and remove unimportant information. This has applications in information theory, data science, and machine learning. We'll talk more about this in EECS16B.

EECS 16A Designing Information Devices and Systems I
Spring 2021 Discussion 5B

## 1. Diagonalization

One of the most powerful ways to think about matrices is to think of them in diagonal form [1].

(a) Consider following three matrices: a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and finally a diagonal matrix $\boldsymbol{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$A\vec{v}_i = \lambda_i \vec{v}_i$

$$\mathbf{A}\cancel{\mathbf{V}} = \mathbf{V}\boldsymbol{\Lambda}V^{-1}$$

consider $\lambda_1 \dots \lambda_n$ eigenvalues
$\vec{v}_1 \dots v_n$ eigenvectors

(b) By multiplying both sides on the right by $\mathbf{V}^{-1}$, we get following expression: $\mathbf{A} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}$. This is called the diagonal form of $\mathbf{A}$. Using this expression, we can consider the action of $\mathbf{A}$ on a coordinate vector $\vec{r}^{(u)}$ in the standard basis. Please interpret each step of the following calculation in terms of coordinate transformations and scaling by eigenvalues.

$$\mathbf{A}\vec{r}^{(u)} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^{-1}\vec{r}^{(u)}$$

(a)

$$AV = A\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ A\vec{v}_1 & \dots & A\vec{v}_n \\ | & & | \end{bmatrix} = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \dots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix}$$

$$V\Lambda = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \dots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

(b) $A\vec{r}^{(u)} = V\Lambda \underbrace{V^{-1}\vec{r}^{(u)}}_{\vec{r}^{(V)}}$

going into the V basis

some transform, scaling the matrix

out of the V basis, back to std basis

$$\begin{tikzcd} \vec{r} \arrow[r, "\text{into } V \text{ basis}"] \arrow[d, "\text{transform in std basis}"'] & V^{-1}\vec{r} \arrow[d, "\text{transform in } V \text{ basis}"] \\ A\vec{r} & \Lambda V^{-1}\vec{r} \arrow[l, "\text{out of } V \text{ basis}"] \end{tikzcd}$$

[1] Not all matrices can be put in this form but most can. The ones that can't be diagonalized can be put in a similar form called the Jordan form.

## 2. Fibonacci Sequence

***Learning Goal:*** *This problem uses the famous Fibonacci sequence to show how writing a matrix in terms of its eigenvalues and eigenvactors can be used to analyze the matrix raised to a power.*

(a) The Fibonacci sequence can be constructed according to the following relation. The $N$th number in the Fibonacci sequence, $F_N$, is computed by adding the previous two numbers in the sequence together:

$$F_N = F_{N-1} + F_{N-2}$$

We select the first two numbers in the sequence to be $F_1 = 0$ and $F_2 = 1$ and then we can compute the following numbers as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Explicitly write down the matrix $\mathbf{A}$ and its elements that can be used to compute the next value of the Fibonacci sequence, $F_N$, from the previous two values, $F_{N-1}$ and $F_{N-2}$.

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

$F_N = F_{N-1} + F_{N-2}$

$F_{N-1} = 1(F_{N-1}) + 0(F_{N-2})$

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

(b) Construct a matrix $\mathbf{P}$ whose columns are the eigenvectors of $\mathbf{A}$, and a matrix $\mathbf{D}$ whose diagonal elements are the corresponding eigenvalues of $\mathbf{A}$.

**P** will have the form $\begin{bmatrix} | & | \\ \vec{p}_1 & \vec{p}_2 \\ | & | \end{bmatrix}$ where $\vec{p}_1$ and $\vec{p}_2$ are the first and second eigenvectors of $\mathbf{A}$.

**D** will have the form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where $\lambda_1$ and $\lambda_2$ are the first and second eigenvalues of $\mathbf{A}$.

Diagonalization

Show your work and confirm that $\mathbf{A} = \mathbf{PDP}^{-1}$. (No need to provide a proof).

(c) **(Challenge):** Consider now how to compute $\mathbf{A}^N$ using the result that $\mathbf{A} = \mathbf{PDP}^{-1}$. We can write $\mathbf{A}^N = (\mathbf{PDP}^{-1})^N = \mathbf{PD}^N\mathbf{P}^{-1}$. Can you justify that? Use that equation to confirm that

$$F_N = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{N-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{N-1}$$

is an analytical expression for the $N$th Fibonacci number.

$$A^N = (PDP^{-1})^N = PD\underbrace{\cancel{P^{-1}P}}_{I}D\underbrace{\cancel{P^{-1}P}}_{I}D\underbrace{P^{-1}}_{I} \ldots \underbrace{P}_{I}DP^{-1}$$

$$= P\underbrace{D \ldots D}_{N}P^{-1} = PD^NP^{-1}$$

$\vec{r} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 \ldots$

$= r_1^{(v)} \vec{v}_1 + r_2^{(v)} \vec{v}_2 \ldots$

$$F_N = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix}$$

$$D^N = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{bmatrix}^N = \begin{bmatrix} \lambda_1^N & & 0 \\ & \lambda_2^N & \\ 0 & & \ddots \lambda_N^N \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} A^{N-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} P D^{(N-2)} P^{-1} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$$

  

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} P \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^N & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^N \end{bmatrix} P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

(1) Found eigenvalues

$$\det(A - \lambda I) = (1-\lambda)(-\lambda) - 1 = 0$$

$$= \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Eigenvalues $\lambda_1 = \frac{1+\sqrt{5}}{2}$ $\quad \lambda_2 = \frac{1-\sqrt{5}}{2}$

$$\boxed{(1-\sqrt{5})(1+\sqrt{5}) = -4}$$

Eigenvectors

$\lambda_1 = \frac{1+\sqrt{5}}{2}$

(2) Find eigenvectors

$$\left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{2}{1-\sqrt{5}} & 0 \\ 1 & -\frac{1+\sqrt{5}}{2} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{2}{1-\sqrt{5}} & 0 \\ 0 & -\frac{1+\sqrt{5}}{2} - \frac{2}{1-\sqrt{5}} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{2}{1-\sqrt{5}} & 0 \\ 0 & \frac{-(1+\sqrt{5})(1-\sqrt{5}) - 4}{2(1-\sqrt{5})} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & \frac{2}{1-\sqrt{5}} & 0 \\ 0 & \cancel{\frac{-1+5-4}{2(1-\sqrt{5})}} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{2}{1-\sqrt{5}} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \vec{v}_1 = \begin{bmatrix} \frac{-2}{1-\sqrt{5}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \qquad \vec{v}_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \vert & \vert \\ \vec{v}_1 & \vec{v}_2 \\ \vert & \vert \end{bmatrix} = \begin{array}{c} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \\ \begin{array}{cc} v_1 & v_2 \end{array} \end{array} \qquad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad ad-bc = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

$$P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \stackrel{?}{=} PDP^{-1} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{3+\sqrt{5}}{2} & \frac{3-\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -\left(\frac{1-\sqrt{5}}{2}\right) \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{5} & -\left(\frac{-2-2\sqrt{5}}{4}\right) + \left(\frac{-2+2\sqrt{5}}{4}\right) \\ \sqrt{5} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \checkmark$$

**3. (PRACTICE) Matrix Powers**

One of the most powerful things about matrix diagonalization is that it gives us some insight into polynomial functions of matrices. Suppose we have some diagonalizable matrix $\mathbf{A}^{n\times n}$:

(a) Write $\mathbf{A}^N$ using the diagonalization of $\mathbf{A}$, simplify the matrix product as much as possible.

(b) Given $\mathbf{A}^N$ in the simplified form as found above, explicitly write out the elements of the expression's matrix product of diagonal eigenvalue matrices $\mathbf{\Lambda}$. What happens to the diagonal elements?

(c) Now that we fully understand the diagonal expansion of $\mathbf{A}^N$, what if I told you that this formula works beyond natural numbers like $N = 1, 2, ...$? Use this formula to express the inverse $\mathbf{A}^{-1}$. Does this agree with what we have seen for inverses of matrix products? And what happens if $\lambda_1 = 0$?

(a) $A^N = (PDP^{-1})^N = \underbrace{PD\overbrace{\cancel{P^{-1}}\cancel{P}}^{I}D\overbrace{\cancel{P^{-1}}}^{I} \cdots \overbrace{\cancel{P}}^{I}DP^{-1}}_{N\times} = PD^NP^{-1}$

(b) $D^N = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix}^N = \begin{bmatrix} \lambda_1^N & & & 0 \\ & \lambda_2^N & & \\ & & \ddots & \\ 0 & & & \lambda_N^N \end{bmatrix}$

$D^N \rightsquigarrow$ diagonals get raised to exponent

(c) $A^{-1} = PD^{-1}P^{-1}$ based on the above

but also $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$

$D^{-1} = \begin{bmatrix} \lambda_1^{-1} & & & 0 \\ & \lambda_2^{-1} & & \\ & & \ddots & \\ 0 & & & \lambda_N^{-1} \end{bmatrix}$

if $\lambda_1 = 0$

$\lambda_1^{-1}$ DNE

so $D^{-1}$ cannot exist

so $A^{-1}$ cannot exist.