Dis 1A EECS16A

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linear algebra: study of linear functions and equations

linear equation: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

coefficients variables constant

Ex: $2x + y = 6$
$3x - 2y = 2$
system of linear equations

$$\left[\begin{array}{cc|c} x & y & \\ 2 & 1 & 6 \\ 3 & -2 & 2 \end{array}\right] \leftarrow \text{augmented matrix}$$

unknowns: x, y

how to solve for unknowns?

- elimination
- substitution
- Gaussian Elimination
  - ↳ 3 row operations
    1. scale
    2. add/subtract
    3. swap

Ex:

	linear
$3x + 4y + 5z = 7$	yes
$2x^2 + y^3 = 0$	no
$3e^x + 1 = 5$	no
$6xy = z$	no
$8x_1 = -5x_2$	yes

3 possible solution cases

① unique sol

② infinite sol

③ no sol

1. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

(a)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

①✓ ②

$$\left[\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ 2 & 0 & 4 & 6 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 2 & 0 & 3 \end{array}\right]$$

③ row2 ①✓ ②✓ ③

$$\xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & -2 & 0 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -6 & 6 \end{array}\right]$$

row3 ①✓ ② ⑥

$$\xrightarrow{R_3 \leftarrow \frac{R_3}{-6}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

ref

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

rref

$1x_1 + 0x_2 + 0x_3 = 5$
$0x_1 + 1x_2 + 0x_2 = -1$
$0x_1 + 0x_2 + 1x_3 = -1$

$$\boxed{\begin{aligned} x_1 &= 5 \\ x_2 &= -1 \\ x_3 &= -1 \end{aligned}}$$

unique sol

Gaussian Elimination Algorithm

① make sure row 1 has a nonzero coefficient for x_1 (otherwise swap rows)
② normalize so x_1 has coefficient of 1
③ use row 1 to eliminate x_1 from rows below
④ repeat with next row
⑤ get to an upper triangular matrix ref
⑥ back-substitute (zeros above pivots)

row echelon form: (ref) pivot is always to the right of the pivot of the row above & rows of all zeros are at bottom

🟩 pivot: leading coefficient/entry
left-most nonzero number in a row

row reduced echelon form: (rref)
ref + zeros above pivots

(b)

$$\begin{bmatrix}1&4&2\\1&2&8\\1&3&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\3\end{bmatrix}$$

$x + y = 1$
$x + y = 2$

row1
① ✓
② ✓

$$\left[\begin{array}{ccc|c}1&4&2&2\\1&2&8&0\\1&3&5&3\end{array}\right]\xrightarrow[R_2\leftarrow R_2-R_1]{③}\left[\begin{array}{ccc|c}1&4&2&2\\0&-2&6&-2\\1&3&5&3\end{array}\right]\xrightarrow{R_3\leftarrow R_3-R_1}\left[\begin{array}{ccc|c}1&4&2&2\\0&-2&6&-2\\0&-1&3&1\end{array}\right]$$

row 2
① ✓ ②

$$\xrightarrow{R_2\leftarrow \frac{R_2}{-2}}\left[\begin{array}{ccc|c}1&4&2&2\\0&1&-3&1\\0&-1&3&1\end{array}\right]\xrightarrow[R_3\leftarrow R_3+R_2]{③}\underset{\text{ref}}{\left[\begin{array}{ccc|c}1&4&2&2\\0&1&-3&1\\0&0&0&2\end{array}\right]}$$

$0\cdot x_1 + 0x_2 + 0x_3 = 2$
$0 \neq 2$

no sol

(c)

$$\begin{bmatrix}2&2&3\\0&1&1\\2&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}7\\3\\1\end{bmatrix}$$

$x + y = 1$
$2x + 2y = 2$

$$\left[\begin{array}{ccc|c}2&2&3&7\\0&1&1&3\\2&0&1&1\end{array}\right]\xrightarrow{R_1\leftarrow \frac{R_1}{2}}\left[\begin{array}{ccc|c}1&1&\frac{3}{2}&7/2\\0&1&1&3\\2&0&1&1\end{array}\right]\xrightarrow{R_3\leftarrow R_3-R_1}\left[\begin{array}{ccc|c}1&1&\frac{3}{2}&7/2\\0&1&1&3\\0&-2&-2&-6\end{array}\right]$$

$$\xrightarrow{R_3\leftarrow R_3+2R_2}\underset{\text{ref}}{\left[\begin{array}{ccc|c}1&1&\frac{3}{2}&7/2\\0&1&1&3\\0&0&0&0\end{array}\right]}\xrightarrow{R_1\leftarrow R_1-R_2}\underset{\text{rref}}{\left[\begin{array}{ccc|c}1&0&-1/2&1/2\\0&1&1&3\\0&0&0&0\end{array}\right]}$$

inf sol

$x_1 - 1/2\,x_3 = 1/2$
$x_2 + x_3 = 3$

$x_3 = 0 \rightarrow x_1 = \frac{1}{2}\quad x_2 = 3$
$x_3 = 1 \rightarrow x_1 = 1\quad x_2 = 2$

$$\left.\begin{array}{l}x_1 = 1/2\,x_3 + 1/2\\x_2 = 3 - x_3\end{array}\right\}$$

✓ ✗

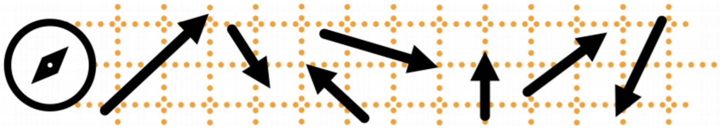
(d) True or False: A system of equations with more equations than unknowns will always have either infinite solutions or no solutions.

$x - y = 0$
$x + y = 2$
$2x - 2y = 0$

more eqns
than
unknowns
→ unique

$$\left[\begin{array}{cc|c}1&-1&0\\1&1&2\\2&-2&0\end{array}\right]\longrightarrow\left[\begin{array}{cc|c}1&0&1\\0&1&1\\0&0&0\end{array}\right]$$

2. Vectors



A vector is an ordered list of numbers. For instance, a point on a plane (x,y) is a vector! We label vectors using an arrow overhead $\vec{v}$, and since vectors can live in ANY dimension of space we'll need to leave our notation general $(x,y) \rightarrow \vec{v} = (v_1, v_2, \ldots)$. Below are few more examples (the left-most form is the general definition):

vector

in

dimension of space

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \vec{a} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3 \qquad \vec{b} = \begin{bmatrix} 2.4 \\ 5.3 \end{bmatrix} \in \mathbb{R}^2$$

real

Just to unpack this a bit more, $\vec{b} \in \mathbb{R}^3$ in english means "vector $\vec{b}$ lives in 3-Dimensional space".

- The $\in$ symbol literally means "in"
- The $\mathbb{R}$ stands for "real numbers" (FUN FACT: $\mathbb{Z}$ means "integers" like $-2, 4, 0, \ldots$)
- The exponent $\mathbb{R}^{n} \leftarrow$ indicates the dimension of space, or the amount of numbers in the vector.

One last thing: it is standard to write vectors in column-form, like seen with $\vec{a}, \vec{b}, \vec{x}$ above. We call these *column vectors*, in contrast to horizontally written vectors which we call *row vectors*.

column $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ row $\begin{bmatrix} 1 & 2 \end{bmatrix}$

Okay, let's dig into a few examples:

(a) Which of the following vectors live in $\mathbb{R}^2$ space?

i. $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ yes

ii. $\begin{bmatrix} 5 \\ 0 \\ 3 \\ 5 \end{bmatrix}$ no

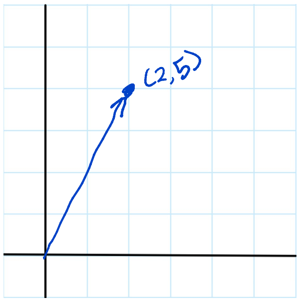
iii. $\begin{bmatrix} -4.76 \\ 1.32 \\ 0.01 \end{bmatrix}$ no

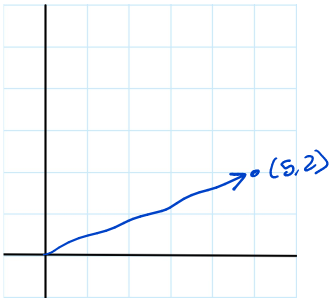
iv. $\begin{bmatrix} -20 \\ 100 \end{bmatrix}$ yes

$\mathbb{R}^2$ $\vec{x}$

(b) Graphically show the vectors (either in a sketch with axes, or a plot on a computer):

i. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (2,5) ii. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$





(c) Compute the sum $\vec{a} + \vec{b} = \vec{c}$ from the vectors below, and then graphically sketch or plot these vectors. (show them in a way that forms a triangle; also is there only one possible triangle?)

$$\vec{c} = \vec{a} + \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$\vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$