Dis 2A EECS16A

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Review

linear combination: $\underbrace{\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n}$ $= \vec{b}$

scalar coefficients (α_i) — vectors ($\vec{v}_i$)

" $\vec{b}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_n$ "

Ex: $\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}$ $\quad \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$

$\begin{bmatrix}1\\2\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\end{bmatrix}$

Ex: $\vec{b} = \begin{bmatrix}2\\0\end{bmatrix}$

$\begin{bmatrix}2\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\end{bmatrix}$

$\alpha_1 = 2$

Ex: is $\vec{b} = \begin{bmatrix}1\\2\end{bmatrix}$ a linear combination of $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$?

i.e. does there exist some α_1, α_2 such that

$$\begin{bmatrix}1\\2\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}$$

yes! $\alpha_1 = 1$, $\alpha_2 = 2$

$\left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & 2\end{array}\right]$

$\vec{b}$ is a linear combo of $\vec{v}_1, \vec{v}_2$

def of **span**: for a set of vectors $\vec{v}_1, \ldots, \vec{v}_n$

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \Big\{\underbrace{\sum_{i=1}^{n} \alpha_i \vec{v}_i}_{\text{linear combination of } \vec{v}_1, \ldots, \vec{v}_n} \;\Big|\; \alpha_i \in \mathbb{R}\Big\}$$

scalar real numbers

"all possible linear combinations of $\vec{v}_1, \ldots, \vec{v}_n$"

Ex: $\vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \underbrace{\alpha_1\begin{bmatrix}1\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\end{bmatrix}} = \begin{bmatrix}*\\*\end{bmatrix}$

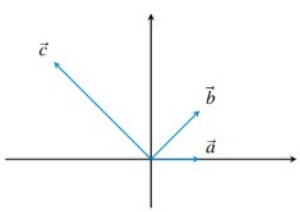
$\alpha_1 = 1$, $\alpha_2 = 1$ $\quad$ $\alpha_1 = 18$, $\alpha_2 = 300$

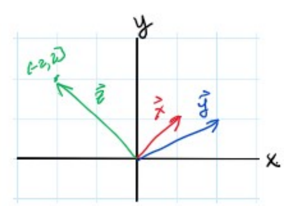
→ inf combinations of α_1, α_2

→ span all of $\mathbb{R}^2$ describes every 2-dimensional vector

1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a} + \beta\vec{b} = \vec{c}$.





(a) First, consider the case where $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$, $\vec{y} = \begin{bmatrix}2\\1\end{bmatrix}$, and $\vec{z} = \begin{bmatrix}-2\\2\end{bmatrix}$. Draw these vectors on a sheet of paper.

(b) We want to find the two scalars α and β, such that by moving α along $\vec{x}$ and β along $\vec{y}$ so that we can reach $\vec{z}$. Write a system of equations to find α and β in matrix form.

$\vec{z} = \alpha\vec{x} + \beta\vec{y}$

$$\begin{bmatrix}-2\\2\end{bmatrix} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\alpha\end{bmatrix} + \begin{bmatrix}2\beta\\\beta\end{bmatrix} = \begin{bmatrix}\alpha + 2\beta\\\alpha + \beta\end{bmatrix} \rightarrow \left.\begin{aligned}-2 &= \alpha + 2\beta\\2 &= \alpha + \beta\end{aligned}\right\}$$

$$\left[\begin{array}{cc|c}1 & 2 & -2\\1 & 1 & 2\end{array}\right]$$

(c) Solve for α, β.

$$\left[\begin{array}{cc|c}1 & 2 & -2\\1 & 1 & 2\end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{cc|c}1 & 2 & -2\\0 & -1 & 4\end{array}\right] \xrightarrow{R_2 \leftarrow \frac{R_2}{-1}} \left[\begin{array}{cc|c}1 & 2 & -2\\0 & 1 & -4\end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{cc|c}1 & 0 & 6\\0 & 1 & -4\end{array}\right]$$

$\alpha = 6$
$\beta = -4$

* unique

$\text{span}\{\vec{x}, \vec{y}\} = \alpha\vec{x} + \beta\vec{y}$

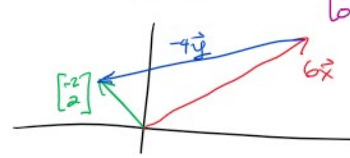
$\vec{z} = 6\vec{x} - 4\vec{y}$

$\vec{z}$ is a lin combo of $\vec{x}, \vec{y}$ → $\vec{z} \in \text{span}\{\vec{x}, \vec{y}\}$

Ex: $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ $\vec{y} = \begin{bmatrix}2\\0\end{bmatrix}$

$\vec{z} = \begin{bmatrix}3\\3\end{bmatrix}$

$\begin{bmatrix}3\\0\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}2\\0\end{bmatrix}$



2. Span basics

(a) What is span $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$?

$= \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix}$ & for any α_1, α_2

Ex: is $\begin{bmatrix}0\\0\\1\end{bmatrix}$ in span $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$?

$\begin{bmatrix}0\\0\\1\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix}$

can we find α_1, α_2? no

(b) Is $\begin{bmatrix}5\\5\\0\end{bmatrix}$ in span $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix} \right\}$?

$\begin{bmatrix}5\\5\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}1&2&5\\2&1&5\\0&0&0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1&0&5/3\\0&1&5/3\\0&0&0\end{array}\right]$

can find α_1, α_2

yes it is in the span

$\alpha_1 = \frac{5}{3}$ $\alpha_2 = \frac{5}{3}$ unique

(c) What is a possible choice for $\vec{v}$ that would make span $\left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \vec{v} \right\} = \mathbb{R}^3$?

$\vec{v} = \begin{bmatrix}0\\0\\1\end{bmatrix}$ or $\begin{bmatrix}1\\2\\3\end{bmatrix}$ or $\begin{bmatrix}1\\0\\-1\end{bmatrix}$

$\begin{bmatrix}*\\*\\*\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}2\\1\\0\end{bmatrix} + \alpha_3 \begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}$

$\left[\begin{array}{ccc|c}1&2&v_1&*\\2&1&v_2&*\\0&0&v_3&*\end{array}\right]$ must have a unique soln

$v_3 \neq 0$

(d) For what values of b_1, b_2, b_3 is the following system of linear equations consistent? ("Consistent" means there is at least one solution.)

unique or inf

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$\begin{bmatrix}1&2\\2&1\\0&0\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

$\vec{b} = x_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + x_2 \begin{bmatrix}2\\1\\0\end{bmatrix}$

$\left[\begin{array}{cc|c}1&2&b_1\\2&1&b_2\\0&0&b_3\end{array}\right]$

b_1, b_2 can be anything

$b_3 = 0$

Now let's look at the iPython demo!

3. Span Proofs

Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

(a)

$$\underbrace{\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}}_{\text{span 1}} = \underbrace{\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}}_{\text{span 2}}, \text{ where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

$\vec{q} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$

for some scalars a_i

$\vec{r} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$

for some scalars b_i

$\vec{q} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = \left(\frac{a_1}{\alpha}\right)(\alpha\vec{v}_1) + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$

$a_1\vec{v}_1 = \frac{a_1}{\cancel{\alpha}}\cancel{\alpha}\vec{v}_1$

$\vec{q} \in \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\vec{r} = b_1(\alpha\vec{v}_1) + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n = (b_1\alpha)\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_n\vec{v}_n$

$\vec{r} \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\alpha\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

Strategies for Proofs

- write out mathematical defn for what you know & what you want to show
- try simple examples to find patterns
- manipulate the defns to get from what you know to what you want to show

Ex: span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} \stackrel{?}{=}$ span $\left\{ \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

$\begin{bmatrix}1\\2\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\0\end{bmatrix} = 1 \cdot \begin{bmatrix}1\\0\\0\end{bmatrix} + 2 \begin{bmatrix}0\\1\\0\end{bmatrix}$

$\begin{bmatrix}1\\2\\0\end{bmatrix} = \alpha_1 \begin{bmatrix}2\\0\\0\end{bmatrix} + \alpha_2 \begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{1}{2} \begin{bmatrix}2\\0\\0\end{bmatrix} + 2 \begin{bmatrix}0\\1\\0\end{bmatrix}$

Ex: $\begin{bmatrix}3\\0\\0\end{bmatrix} =$ span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$

$\begin{bmatrix}3\\0\\0\end{bmatrix} \in$ span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \subseteq$ span $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$

subset

similar to less than or equal to

(b) **(Practice)**

$$\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \text{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.