Dis 2B EECS16A

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Matrix Multiplication

dimensions of a matrix:

$\# \text{rows} \times \# \text{cols}$

PQ

$m\times n \quad p \times q$

$n=p$ to have valid matrix mult product

$$\underbrace{\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}}_{\mathbf{B}} = \underbrace{\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22}\\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{bmatrix}}_{\mathbf{AB}}$$

2×2

1. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A}=\begin{bmatrix}1 & 4\end{bmatrix} \quad \mathbf{B}=\begin{bmatrix}3\\2\end{bmatrix} \quad \mathbf{C}=\begin{bmatrix}1&4\\2&3\end{bmatrix} \quad \mathbf{D}=\begin{bmatrix}3&2\\2&1\end{bmatrix}$$

$$\mathbf{E}=\begin{bmatrix}1&9&5&7\\4&3&2&2\end{bmatrix} \quad \mathbf{F}=\begin{bmatrix}5&5&8\\6&1&2\\4&1&7\\3&2&2\end{bmatrix} \quad \mathbf{G}=\begin{bmatrix}8&1&6\\3&5&7\\4&9&2\end{bmatrix} \quad \mathbf{H}=\begin{bmatrix}5&3&4\\1&8&2\\2&3&5\end{bmatrix}$$

For each matrix multiplication problem, *if the product exists*, find the product by hand. Otherwise, explain why the product does not exist.

(a) **A B**

$1\times 2 \quad 2\times 1$

$$\begin{bmatrix}1 & 4\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = 1\cdot 3 + 4\cdot 2 = 11$$

1×1

(b) **C D**

$2\times 2 \quad 2\times 2$

$$\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}3&2\\2&1\end{bmatrix} = \begin{bmatrix}1\cdot 3+4\cdot 2 & 1\cdot 2+4\cdot 1\\ 2\cdot 3+3\cdot 2 & 2\cdot 2+3\cdot 1\end{bmatrix} = \begin{bmatrix}11 & 6\\12 & 7\end{bmatrix}$$

not the same!

(c) **D C**

$2\times 2 \quad 2\times 2$

$$\begin{bmatrix}3&2\\2&1\end{bmatrix}\begin{bmatrix}1&4\\2&3\end{bmatrix} = \cancel{\begin{bmatrix}11 & 6\\12 & 7\end{bmatrix}} = \begin{bmatrix}7 & 18\\4 & 11\end{bmatrix}$$

matrix multiplication is not commutative!

$CD \neq DC$

(d) **C E** $= 2\times 4$

$2\times 2 \quad 2\times 4$

$2=2$ ✓

$$\begin{bmatrix}1&4\\2&3\end{bmatrix}\begin{bmatrix}1&9&5&7\\4&3&2&2\end{bmatrix} = \begin{bmatrix}1\cdot 1+4\cdot 4 & 1\cdot 9+4\cdot 3 & \cdots\\ 2\cdot 1+3\cdot 4 & 2\cdot 9+3\cdot 3 & \end{bmatrix} = \begin{bmatrix}17&21&13&15\\14&27&16&20\end{bmatrix}$$

(e) **F E** (only note whether or not the product exists)

$4\times 3 \quad 2\times 4$

no product does not exist $\quad 3\neq 2$

(f) **E F** (only note whether or not the product exists)

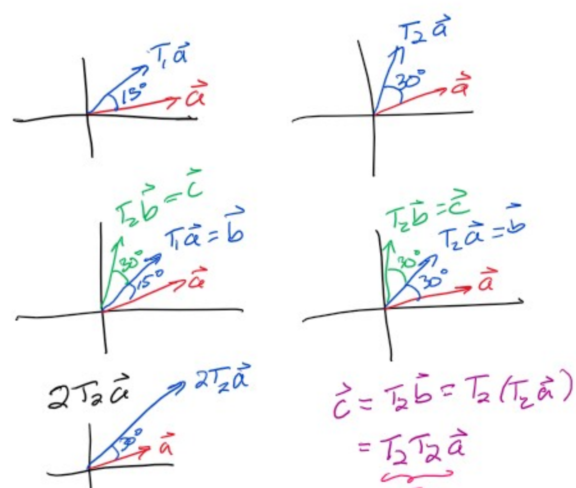
$2\times 4 \quad 4\times 3 \quad 4=4$

yes product does exist :)

2. Visualizing Matrices as Operations

This problem is going to help you visualize matrices as operations. For example, when we multiply a vector by a "rotation matrix," we will see it "rotate" in the true sense here. Similarly, when we multiply a vector by a "reflection matrix," we will see it be "reflected." The way we will see this is by applying the operation to all the vertices of a polygon and seeing how the polygon changes.

Your TA will now show you how a unit square can be rotated, scaled, or reflected using matrices!



$\vec{c} = T_2\vec{b} = T_2(T_2\vec{a})$

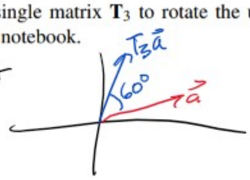
$= \underbrace{T_2T_2}_{T_3}\vec{a}$

Part 1: Rotation Matrices as Rotations

(a) We are given matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, and we are told that they will rotate the unit square by 15° and 30°, respectively. Suggest some methods to rotate the unit square by 45° using only $\mathbf{T}_1$ and $\mathbf{T}_2$. How would you rotate the square by 60°? Your TA will show you the result in the iPython notebook.

(b) Find a single matrix $\mathbf{T}_3$ to rotate the unit square by 60°. Your TA will show you the result in the iPython notebook.

want



$T_3 = T_2T_2$

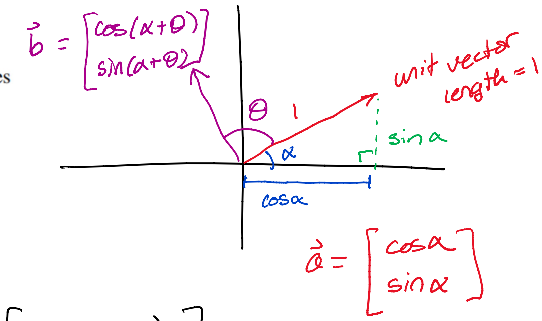
$T_3 = T_2T_1T_1$

$T_3 = T_1T_1T_2$

(c) $\mathbf{T}_1$, $\mathbf{T}_2$, and the matrix you used in part (b) are called “rotation matrices.” They rotate any vector by an angle θ. Show that a rotation matrix has the following form:

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where θ is the angle of rotation. To do this consider rotating the unit vector $\begin{bmatrix} cos(\alpha) \\ sin(\alpha) \end{bmatrix}$ by θ degrees using the matrix $\mathbf{R}$.



(**Definition:** A vector, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \end{bmatrix}$, is a unit vector if $\sqrt{v_1^2 + v_2^2 + \ldots} = 1$.)

(*Hint: Use your trigonometric identities!*)

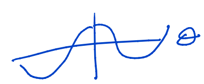
want to show: $\vec{b} = R\vec{a}$ ✓

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\theta - \sin\alpha\sin\theta \\ \cos\alpha\sin\theta + \sin\alpha\cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\alpha+\theta) \\ \sin(\alpha+\theta) \end{bmatrix}}_{\vec{b}}$$

(d) Now, we want to get back the original unit square from the rotated square in part (b). What matrix should we use to do this? (**Note:** Don’t use inverses! Answer this question using your intuition, we will visit inverses very soon in lecture!)

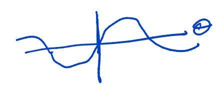
$\theta = -60^\circ$

$$\begin{bmatrix} \cos(-60^\circ) & -\sin(-60^\circ) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$



$\ast$ $\cos(-\theta) = \cos(\theta)$

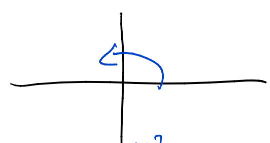
$\sin(-\theta) = -\sin\theta$

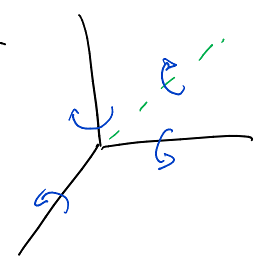


(e) Use part (d) to obtain the “inverse” rotation matrix for a matrix that rotates a vector by θ. Multiply the inverse rotation matrix with the rotation matrix and vice-versa. What do you get?

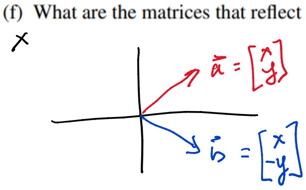
$$R_{inv} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

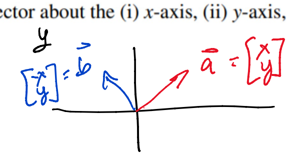
$R \cdot R_{inv}\,\vec{v} = \vec{v}$

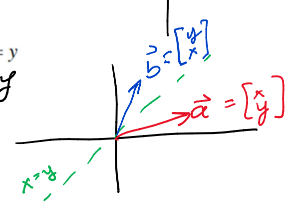




(f) What are the matrices that reflect a vector about the (i) x-axis, (ii) y-axis, and (iii) $x = y$







$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

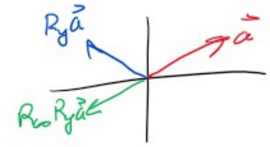
$\ast$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Part 2: Commutativity of Operations

A natural question to ask is the following: Does the *order* in which you apply these operations matter? Your TA will demonstrate parts (a) and (b) in the iPython notebook.

(a) Let's see what happens to the unit square when we rotate the square by 60° and then reflect it along the y-axis.

(b) Now, let's see what happens to the unit square when we first reflect the square along the y-axis and then rotate it by 60°. Is this the same as in part (a)? not the same



(c) Try to do steps (a) and (b) by multiplying the reflection and rotation matrices together (in the correct order for each case). What does this tell you?

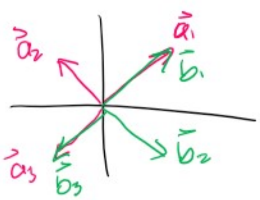
$$R_{60} = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix} \qquad R_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

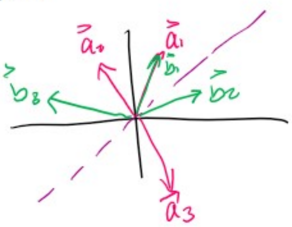
$$\rightarrow R_{60}R_y = \begin{bmatrix} -\cos(60°) & -\sin(60°) \\ -\sin(60°) & \cos(60°) \end{bmatrix}$$ reflected then rotated

not same

$$R_yR_{60} = \begin{bmatrix} -\cos(60°) & \sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix}$$ rotated then reflected

(d) If you reflected the unit square twice (along any pair of axes), do you think the order in which you applied the reflections would matter? Why/why not?





generally order matters

Part 3: Distributivity of Operations

(a) The distributivity property of matrix-vector multiplication holds for any vectors and matrices. Show for general $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$ that $\mathbf{A}(\vec{v}_1 + \vec{v}_2) = \mathbf{A}\vec{v}_1 + \mathbf{A}\vec{v}_2$.

$$A(\vec{v}_1 + \vec{v}_2) = A\vec{v}_1 + A\vec{v}_2$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_{11} + v_{21} \\ v_{12} + v_{22} \end{bmatrix} = \ldots \quad = A\vec{v}_1 + A\vec{v}_2$$