Dis 3A EECS16A

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1. Inverses

In general, the *inverse* of a matrix "undoes" the operation that a matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

$I\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}$

$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$

$(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$ for a nonzero scalar $k \in \mathbb{R}$

$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ T is "Transpose"

$(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ assuming $\mathbf{A}, \mathbf{B}$ are both invertible

scalar

$a \cdot a^{-1} = \frac{a}{a} = 1$

why is this useful?

$A\vec{x} = \vec{b}$ ← GE

$A^{-1}(A\vec{x} = \vec{b})$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$ ← matrix mult

* only works if $\vec{x}$ is a unique sol

(a) Suppose $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are all invertible matrices.
Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

know: A, B, C are invertible

$D^{-1}D = DD^{-1} = I$

show: $(ABC)^{-1}(ABC) = I$

$C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC$

$= C^{-1}B^{-1}BC$

$= C^{-1}IC$

$= C^{-1}C$

$= I$ ✓

$CD \neq DC$

$ABC \neq BAC$

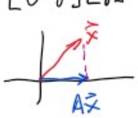
(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?
(*Bonus:* How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)
v. Are all square matrices invertible?
vi. (PRACTICE) How can you find the inverse of a general $n \times n$ matrix?

$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$



ii. $A\vec{x} = \vec{b}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ no

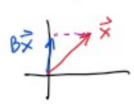
$\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ .15 \end{bmatrix}$

$x = 2$

$0 \cdot x + 0 \cdot y = 0$ $0 = 0$

$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}$



$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\vec{x} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ no

$\vec{x} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} -8 \\ 4 \end{bmatrix}$

$0 = 0$

→ $y = 4$

$\mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \end{bmatrix}$

averaging x, y

$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\vec{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ no

$\vec{x} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ or $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

→ $\frac{1}{2}x + \frac{1}{2}y = 4$

$\frac{1}{2}x + \frac{1}{2}y = 4$

$\mathbf{D} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix}$

weighted combo of x, y

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\vec{x} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$ no

$\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$

$x + 2y = 6$

$2x + 4y = 12$

iii. none of these matrices are invertible

iv. linearly dependent rows & cols * one row is a multiple of the other row ↳ eqns

v. no, see A, B, C, D

vi. Gauss-Jordan method

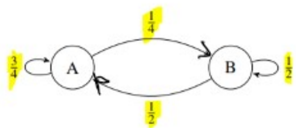
* only square matrices w/ lin ind cols/rows are invertible ↳ unique sol

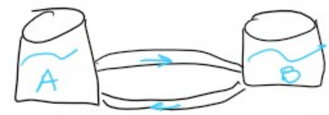
$$AA^{-1} = I \qquad [A \mid I] \xrightarrow[\text{reduce}]{\text{row}} [I \mid A^{-1}]$$

$$A\vec{x} = \vec{b} \qquad [A \mid \vec{b}] \rightarrow [I \mid \vec{x}]$$

2. Transition Matrix

Suppose we have a network of pumps as shown in the diagram below. Let us describe the state of A and B using a state vector $\vec{x}[n] = \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$ where $x_A[n]$ and $x_B[n]$ are the states at time-step n.





	A	B
initially $n=0$	4L	4L
$n=1$	3L+2L = 5L	1L+2L = 3L

$$\vec{x}[0] = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \vec{x}[1] = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

(a) Find the state transition matrix S, such that $\vec{x}[n+1] = S\,\vec{x}[n]$.
Separately find the sum of the terms for each column vector in S. Do you notice any pattern?

$x_A[n]$ ← amount of water in tank A @ time step n

$x_B[n]$ ← " B "

$$x_A[n+1] = \frac{3}{4}x_A[n] + \frac{1}{2}x_B[n]$$
$$x_B[n+1] = \frac{1}{4}x_A[n] + \frac{1}{2}x_B[n]$$
$$\rightarrow \begin{bmatrix} x_A[n+1] \\ x_B[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}}_{S} \begin{bmatrix} x_A[n] \\ x_B[n] \end{bmatrix}$$

* note that cols sum to 1
→ conservative system
→ 100% of water is accounted for
→ no water is being taken or added to system

(b) Let us now find the matrix S^{-1} such that we can recover the previous state $\vec{x}[n-1]$ from $\vec{x}[n]$.
Specifically, solve for S^{-1} such that $\vec{x}[n-1] = S^{-1}\,\vec{x}[n]$.

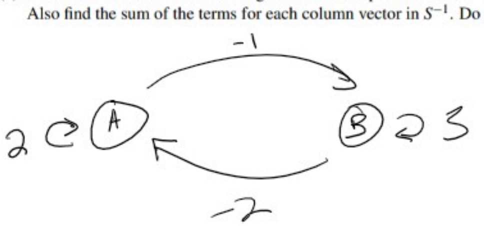
$$[A \mid I] \xrightarrow[\text{reduce}]{\text{row}} [I \mid A^{-1}]$$

$$\left[\begin{array}{cc|cc} \frac{3}{4} & \frac{1}{2} & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{4}{3} & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{4}R_1} \left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow 3R_2} \underbrace{\left[\begin{array}{cc|cc} 1 & \frac{2}{3} & \frac{4}{3} & 0 \\ 0 & 1 & -1 & 3 \end{array}\right]}_{\text{ref}} \xrightarrow{R_1 \leftarrow R_1 - \frac{2}{3}R_2} \underbrace{\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right]}_{\text{rref}}$$

$$\boxed{S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}}$$

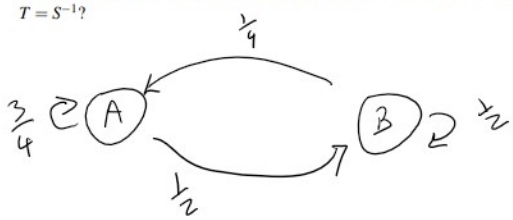
(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.
Also find the sum of the terms for each column vector in S^{-1}. Do you notice any pattern?



$$x_A[n-1] = 2x_A[n] - 2x_B[n]$$
$$x_B[n-1] = -x_A[n] + 3x_B[n]$$

* negative values and values > 1 don't have physical meanings

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?



$$T = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad T \neq S^{-1}$$

↑ actually $T = S^T$ ← switch cols & rows

(e) Suppose we start in the state $\vec{x}[1] = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$. Compute the state vector after 2 time-steps $\vec{x}[3]$.

$\star$ $\vec{x}[n+1] = S\vec{x}[n]$

2 ways

(1) $\vec{x}[2] = S\vec{x}[1] = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 12 \\ 12 \end{bmatrix} = \begin{bmatrix} 9+6 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 15 \\ 9 \end{bmatrix}$

$\vec{x}[3] = S\vec{x}[2] = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 15 \\ 9 \end{bmatrix} = \begin{bmatrix} \frac{45}{4} + \frac{9}{2} \\ \frac{15}{4} + \frac{9}{2} \end{bmatrix} = \begin{bmatrix} 63/4 \\ 33/4 \end{bmatrix}$

(2) $\vec{x}[3] = S\vec{x}[2] = S(S\vec{x}[1])$

$= SS\vec{x}[1] = S^2\vec{x}[1]$

(f) **(Challenge practice problem)** Given our starting state from the previous problem, what happens if we look at the state of the network after a lot of time steps? Specifically which state are we approaching, as defined below?

$$\vec{x}_{final} = \lim_{n\to\infty} \vec{x}[n]$$

Note that the final state needs to be what we call a *steady state*, meaning $S\,\vec{x}_{final} = \vec{x}_{final}$.

Also what can you say about $x_A[n] + x_B[n]$?

Use information from both of these properties to write out a new system of equations and solve for $\vec{x}_{final}$.

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