Dis 3B EECS16A

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Gauss-Jordan method $\quad AA^{-1} = I \quad [A \mid I] \rightarrow [I \mid A^{-1}]$

$A\vec{x} = \vec{b} \quad [A \mid \vec{b}] \rightarrow [I \mid \vec{x}]$

1. Mechanical Inverses

For each sub-part below, determine whether or not the inverse of $\mathbf{A}$ exists. If it exists, compute the inverse using Gauss-Jordan method.

matrix is not invertible if cols/rows are lin dep

" " if it is not square

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 9 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{R_2}{9}} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{9} \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{9} \end{bmatrix}$$

(b) (PRACTICE)

$\mathbf{A} = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{R_1}{a}} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - cR_1} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{d - \frac{bc}{a}} R_2} \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-\frac{c}{a}}{d - \frac{bc}{a}} \cdot \frac{a}{a} & \frac{1}{d - \frac{bc}{a}} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - \frac{b}{a} R_2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} + \frac{b}{a}\frac{c}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\boxed{A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

(d) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 3 \\ 2 & -2 & 4 \end{bmatrix}$ $\quad$ not invertible $\rightarrow$ not square

(e) $\mathbf{A} = \begin{bmatrix} 5 & 5 & 15 \\ 2 & 2 & 4 \\ 1 & 1 & 4 \end{bmatrix}$

$\vec{a}_1 \; \vec{a}_2 \; \vec{a}_3$

$\vec{a}_1 = \vec{a}_2$

$$\left[\begin{array}{ccc|ccc} 5 & 5 & 15 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{5} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & -2 & -2/5 & 1 & 0 \\ 0 & 0 & 1 & -1/5 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2/-2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1/5 & 0 & 0 \\ 0 & 0 & 1 & 1/5 & -1/2 & 0 \\ 0 & 0 & 1 & -1/5 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_2 - R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & \frac{1}{2} & 1 \end{array}\right]$$

row of zeros

= lin dep cols/rows

$\rightarrow$ not invertible

2. Exploring Column Spaces and Null Spaces

- The **column space** is the **span** of the column vectors of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following matrices, answer the following questions:

i. What is the column space of **A**? What is its dimension?
ii. What is the null space of **A**? What is its dimension?
iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same?
iv. Do the columns of **A** span $\mathbb{R}^2$? Do they form a basis for $\mathbb{R}^2$? Why or why not?

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

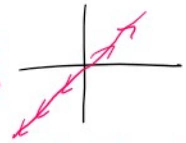
col space: $C(A) = \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$

null space: $N(A)$ = all $\vec{x}$ such that $A\vec{x} = \vec{0}$

dimension of a vector space: number of basis vectors

basis vectors: linearly independent vectors that define a vector space

$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ * exact same vector space



basis: $\begin{bmatrix}1\\1\end{bmatrix}$ or $\begin{bmatrix}-1\\-1\end{bmatrix}$

i. What is the column space of **A**? What is its dimension?

(a) $\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$ $C(A) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ dim = 1 ← only 1 lin ind vector to define space

lin dep col vectors

$= \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\0\end{bmatrix}$

(b) $\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$ $C(A) = \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ dim = 1

(c) $\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix}\underline{1} & 0\\0 & \underline{1}\end{bmatrix}$ lin ind col vectors $C(A) = \text{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\} = \mathbb{R}^2$

max # of pivots = 2

max # of lin ind vec $\in \mathbb{R}^2 = 2$

dim = 2

(d) $\begin{bmatrix}-2 & 4\\3 & -6\end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix}\underline{1} & -2\\0 & 0\end{bmatrix}$ 1 pivot = 1 lin ind col $C(A) = \text{span}\left\{\begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}4\\-6\end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix}-2\\3\end{bmatrix}\right\}$

dim = 1

(e) $\begin{bmatrix}1 & -1 & -2 & -4\\1 & 1 & 3 & -3\end{bmatrix} \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix}\underline{1} & 0 & \frac{1}{2} & -7/2\\0 & \underline{1} & \frac{5}{2} & 1/2\end{bmatrix}$ 2 pivots = 2 lin ind cols

$C(A) = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}-2\\3\end{bmatrix}, \begin{bmatrix}-4\\-3\end{bmatrix}\right\}$

$= \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}\right\} = \mathbb{R}^2$

ii. What is the null space of **A**? What is its dimension?

(a) $\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $N(A) =$ all $\vec{x}$ s.t. $A\vec{x} = \vec{0}$

$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\left[\begin{array}{cc|c}1&0&0\\0&0&0\end{array}\right]$ $x_1 = 0$, $x_2 =$ anything $= \alpha$

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\\alpha\end{bmatrix} = \alpha\begin{bmatrix}0\\1\end{bmatrix}$

$N(A) = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$

dim = 1

$\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0\\-5\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

(b) $\begin{bmatrix}0&1\\0&1\end{bmatrix}$ $\left[\begin{array}{cc|c}0&1&0\\0&1&0\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}0&1&0\\0&0&0\end{array}\right]$ $x_2 = 0$, $x_1 =$ anything $= \alpha$

$0 \cdot x_1 + 0x_2 = 0$ $0 = 0$

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\alpha\\0\end{bmatrix} = \alpha\begin{bmatrix}1\\0\end{bmatrix}$

$N(A) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ dim = 1

(c) $\begin{bmatrix}1&2\\-1&1\end{bmatrix}$ $\left[\begin{array}{cc|c}1&2&0\\-1&1&0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}1&0&0\\0&1&0\end{array}\right]$ $x_1 = 0$, $x_2 = 0$ $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$N(A) = \text{span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \vec{0}$ dim = 0

(d) $\begin{bmatrix}-2&4\\3&-6\end{bmatrix}$ $\left[\begin{array}{cc|c}-2&4&0\\3&-6&0\end{array}\right] \rightarrow \left[\begin{array}{cc|c}\underline{1}&-2&0\\0&0&0\end{array}\right]$ $x_1 - 2x_2 = 0$, $x_2 =$ anything $= \alpha$, $x_1 = 2\alpha$

$\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\alpha\\\alpha\end{bmatrix} = \alpha\begin{bmatrix}2\\1\end{bmatrix}$

$N(A) = \text{span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ $\in \mathbb{R}^2$ dim = 1

$\begin{bmatrix}-2&4\\3&-6\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}1\\1\end{bmatrix} \in \mathbb{R}^2$ $\begin{bmatrix}1\\2\\3\end{bmatrix} \in \mathbb{R}^3$

(e) $\begin{bmatrix}1&-1&-2&-4\\1&1&3&-3\end{bmatrix}$ $N(A) \in \mathbb{R}^4$ $N(A) =$ all $\vec{x}$ s.t. $A\vec{x} = \vec{0}$

2×4 4×1

$\left[\begin{array}{cccc|c}1&-1&-2&-4&0\\1&1&3&-3&0\end{array}\right] \rightarrow \left[\begin{array}{cccc|c}\underline{1}&0&\frac{1}{2}&-7/2&0\\0&\underline{1}&5/2&1/2&0\end{array}\right]$ $x_3 = \alpha$, $x_4 = \beta$ anything

$x_1 + \frac{1}{2}x_3 - \frac{7}{2}x_4 = 0$ $x_1 = -\frac{1}{2}\alpha + \frac{7}{2}\beta$

$x_2 + \frac{5}{2}x_3 + \frac{1}{2}x_4 = 0$ $x_2 = -\frac{5}{2}\alpha - \frac{1}{2}\beta$

$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\alpha + \frac{7}{2}\beta\\-\frac{5}{2}\alpha - \frac{1}{2}\beta\\\alpha\\\beta\end{bmatrix} = \alpha\begin{bmatrix}-1/2\\-5/2\\1\\0\end{bmatrix} + \beta\begin{bmatrix}7/2\\-1/2\\0\\1\end{bmatrix}$

$N(A) = \text{span}\left\{\begin{bmatrix}-1/2\\-5/2\\1\\0\end{bmatrix}, \begin{bmatrix}7/2\\-1/2\\0\\1\end{bmatrix}\right\}$

dim = 2

iii. Are the column spaces of the row reduced matrix **A** and the original matrix **A** the same? $C(A_{rr}) \overset{?}{=} C(A)$

row reduced

(a) $\begin{bmatrix}1&0\\0&0\end{bmatrix} = A$ $A_{rr} = \begin{bmatrix}1&0\\0&0\end{bmatrix} = A$ yes

(b) $\begin{bmatrix}0&1\\0&1\end{bmatrix} = A$ $A_{rr} = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ $C(A) \overset{?}{=} C(A_{rr})$ $C(A) = \text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\} \overset{?}{=} C(A_{rr}) = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

no

(c) $\begin{bmatrix}1&2\\-1&1\end{bmatrix} = A$ $A_{rr} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$ $C(A) \overset{?}{=} C(A_{rr})$ $= \mathbb{R}^2$ yes

(d) $\begin{bmatrix}-2&4\\3&-6\end{bmatrix} = A$ $A_{rr} = \begin{bmatrix}1&-2\\0&0\end{bmatrix}$ $C(A) \neq C(A_{rr})$ no

(e) $\begin{bmatrix}1&-1&-2&-4\\1&1&3&-3\end{bmatrix} = A$ $A_{rr} = \begin{bmatrix}1&0&1/2&-7/2\\0&1&5/2&1/2\end{bmatrix}$ yes both span $\mathbb{R}^2$

iv. Do the columns of **A** form a basis for $\mathbb{R}^2$? Why or why not?

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	cols are lin ind vec	+	cols span $\mathbb{R}^2$	=	basis for $\mathbb{R}^2$
(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	✗		✗		✗
(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	✗		✗		✗
(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$	✓		✓		✓
(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$	✗		✗		✗
(e) $\begin{bmatrix} 1 & -1 & -2 & -4 \\ 1 & 1 & 3 & -3 \end{bmatrix}$	✗		✓		✗