Dis 4B EECS16A

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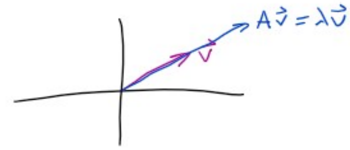
Eigenvalues and Eigenvectors/Eigenspaces   * associated w/ a particular matrix



* $A\vec{v} = \lambda\vec{v}$

eigenvector ($\vec{v}$)   eigenvalue (λ)

* $\vec{v} \neq \vec{0}$

$A\vec{v} - \lambda\vec{v} = \vec{0}$

$A\vec{v} - \lambda I\vec{v} = \vec{0}$

$\underbrace{(A - \lambda I)}_{B}\vec{v} = \vec{0}$

$B\vec{v} = \vec{0}$ where $\vec{v} \neq \vec{0}$

$\vec{v} \in N(B)$ B has a nontrivial nullspace

something in the nullspace besides $\vec{0}$

↳ B has lin dep cols

↳ B is not invertible

↳ $\det(B) = 0$

$\det(A - \lambda I) = 0$ ← use this eqn to solve for λ

$\rightarrow (A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$\vec{v}_1 \in N(A - \lambda_1 I)$

* for a matrix A

if $\det(A) = 0$

A is not invertible

if $\det(A) \neq 0$

A is invertible

Recall from lecture the way to compute a determinant of any 2×2 matrix is by using the following formula:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(\mathbf{A}) = ad - bc$$

1. Mechanical Eigenvalues and Eigenvectors

In each part, find the eigenvalues of the matrix **M** and the associated eigenvectors. State if the inverse of **M** exists.

(a) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\det(M - \lambda I) = 0$

$$M - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$$

$$\det(M - \lambda I) = (-\lambda)(-3-\lambda) - (1)(-2) = 0$$
$$= 3\lambda + \lambda^2 + 2$$
$$= \lambda^2 + 3\lambda - 2$$
$$= (\lambda + 2)(\lambda + 1) = 0$$
$$\rightarrow \lambda_1 = -1 \quad \lambda_2 = -2$$

$M\vec{v}_1 = \lambda_1\vec{v}_1 \rightarrow (M - \lambda_1 I)\vec{v}_1 = \vec{0} \rightarrow \vec{v}_1 \in N(M - \lambda_1 I)$

$$M - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1 \rightarrow R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$ $x_1 = -\alpha$

$x_2 = \text{anything} = \alpha$

$$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\boxed{\text{eigenspace for } \lambda_1 = -1 \text{ is span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}}$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = (-1)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = (-1)\begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

invertible

$\lambda_2 = -2$ . . . eigenspace for $\lambda_2 = -2$ is span$\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}$

(b) $\mathbf{M} = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$ * notice cols are lin dep → what does that mean?

$\det(M - \lambda I) = 0$

Nontrivial nullspace ⇒ eigenvalue of $\lambda = 0$

$$\det\left(\begin{bmatrix} -2-\lambda & 4 \\ -4 & 8-\lambda \end{bmatrix}\right) = (-2-\lambda)(8-\lambda) - (-4)(4)$$
$$= \lambda^2 - 6\lambda - 16 + 16$$
$$= \lambda^2 - 6\lambda$$
$$= \lambda(\lambda - 6) = 0$$
$$\lambda_1 = 0 \quad \lambda_2 = 6$$

$M\vec{x} = \vec{0}$

$\begin{bmatrix} -2 & 4 & | & 0 \\ -4 & 8 & | & 0 \end{bmatrix} \to \cdots$

$N(M) = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

$\lambda_1 = 0$

$(M - \cancel{\lambda_1}^{0} I)\vec{v}_1 = \vec{0}$ $\quad M\vec{v}_1 = \vec{0}$ $\quad \vec{v}_1 \neq \vec{0}$ $\quad \vec{v}_1 \in N(M)$

$\begin{bmatrix} -2 & 4 & | & 0 \\ -4 & 8 & | & 0 \end{bmatrix} \longrightarrow \cdots \longrightarrow \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ $\quad x_1 - 2x_2 = 0 \quad x_1 = 2\alpha$

$x_2 = \text{anything} = \alpha$

$\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad$ eigenspace for $\lambda_1 = 0$ is $\text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} = N(M)$

$\begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

not invertible

$\lambda_2 = 6$ $\quad$ eigenspace for $\lambda_2 = 6$ is $\text{span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$

(c) $\mathbf{M} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\det(M - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}\right) = \lambda^2 - (0)(1) = 0$

$\lambda^2 = 0 \quad \lambda_1 = \lambda_2 = 0$

Ex: $\lambda^2 = 1$

$\lambda_1 = \lambda_2 = 1$

$\lambda = 0 \quad (M - \cancel{\lambda}^{0} I)\vec{v} = M\vec{v} = \vec{0}$

$\begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ $\quad x_2 = 0$, $\quad x_1 = \text{anything} = \alpha$ $\quad \vec{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenspace $\lambda = 0$ is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = N(M)$

not invertible

2. Eigenvalues and Special Matrices – Visualization

An eigenvector $\vec{v}$ belonging to a square matrix $\mathbf{A}$ is a nonzero vector that satisfies

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

where λ is a scalar known as the **eigenvalue** corresponding to eigenvector $\vec{v}$. Rather than mechanically compute the eigenvalues and eigenvectors, answer each part here by reasoning about the matrix at hand.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigenspace $\mathbb{R}^2$

(a) Does the identity matrix in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

$$I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$I\vec{v} = \vec{v} = (1)\vec{v}$

eigenspace for $\lambda = 1$ is $\mathbb{R}^n$

(b) Does a diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ in $\mathbb{R}^n$ have any eigenvalues $\lambda \in \mathbb{R}$? What are the corresponding eigenvectors?

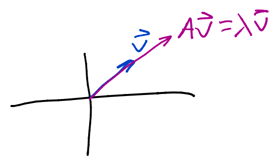
$\lambda_1 = d_1$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\lambda_2 = d_2$ $\quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

$$\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = d_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad \vec{e}_i = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$



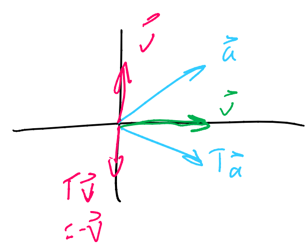
(c) Conceptually, does a rotation matrix in $\mathbb{R}^2$ by angle θ have any eigenvalues $\lambda \in \mathbb{R}$? For which angles is this the case?

$\theta = 180°, 540°, \ldots$ $\quad \lambda = -1$ $\quad R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\theta = 0, 360°, \ldots$ $\quad \lambda = 1$ $\quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) Does the reflection matrix $\mathbf{T}$ across the x-axis in $\mathbb{R}^{2\times2}$ have any eigenvalues $\lambda \in \mathbb{R}$?

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ & diagonal matrix



$\lambda_1 = 1$ $\quad$ eigenspace $= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

$\lambda_2 = -1$ $\quad$ eigenspace $= \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

(f) If a matrix $\mathbf{M}$ has an eigenvalue $\lambda = 0$, what does this say about its null space? What does this say about the solutions of the system of linear equations $\mathbf{M}\vec{x} = \vec{b}$?

→ nullspace is nontrivial

→ inf sol

→ lin dep cols

3. Steady and Unsteady States

(a) You're given the matrix **M**:

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

★ transition matrix

Which generates the next state of a physical system from its previous state: $\vec{x}[k+1] = \mathbf{M}\vec{x}[k]$. ($\vec{x}$ could describe either people or water.) Find the eigenspaces associated with the following eigenvalues:

i. span($\vec{v}_1$), associated with $\lambda_1 = 1$

ii. span($\vec{v}_2$), associated with $\lambda_2 = 2$

iii. span($\vec{v}_3$), associated with $\lambda_3 = \frac{1}{2}$

$M\vec{v}_1 = \lambda_1 \vec{v}_1$

$(M - \lambda I)\vec{v}_1 = \vec{0}$

i. $\left[\begin{array}{ccc|c} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\lambda_1 = 1$

ii.

$\vec{v}_2 = \beta \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$ $\lambda_2 = 2$

$\vec{v}_3 = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\lambda_3 = \frac{1}{2}$

iii.

(b) Define $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$, a linear combination of the eigenvectors. For each of the cases in the table, determine if

$$\lim_{n\to\infty} \mathbf{M}^n \vec{x}[0]$$

converges. If it does, what does it converge to?

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

→ lin ind

→ span $\mathbb{R}^3$

→ any vector in $\mathbb{R}^3$ can be made of a lin combo of these 3 vectors

→ i.e. any $\vec{x}$

what happens way in the future?

$\lim_{n\to\infty} M^n \vec{x}$

$\vec{x}[n+1] = M\vec{x}[n]$

$\vec{x}[1] = M\vec{x}[0]$

$\vec{x}[2] = M\vec{x}[1]$

$= MM\vec{x}[0]$

$= M^2\vec{x}[0]$

$\vec{x}[n] = M^n\vec{x}[0]$

$$\lim_{n\to\infty} M^n \vec{x} = \lim_{n\to\infty} M^n(\alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3)$$

$$= \lim_{n\to\infty} (\alpha M^n\vec{v}_1 + \beta M^n\vec{v}_2 + \gamma M^n\vec{v}_3)$$

$$= \lim_{n\to\infty} (\alpha(\lambda_1)^n\vec{v}_1 + \beta(\lambda_2)^n\vec{v}_2 + \gamma(\lambda_3)^n\vec{v}_3)$$

$$= \lim_{n\to\infty} (\alpha(1)^n\vec{v}_1 + \beta(2)^n\vec{v}_2 + \gamma(\tfrac{1}{2})^n\vec{v}_3)$$

★ $M\vec{v}_1 = \lambda_1\vec{v}_1$

$M^2\vec{v}_1 = MM\vec{v}_1$

$= M(\lambda_1\vec{v}_1)$

$= \lambda_1 M\vec{v}_1$

$= \lambda_1(\lambda_1\vec{v}_1)$

$= \lambda_1^2\vec{v}_1$

α	β	γ	Converges?	$\lim_{n\to\infty}\mathbf{M}^n\vec{x}$
0	0	$\neq 0$	yes	$\vec{0}$
0	$\neq 0$	0	no	/
0	$\neq 0$	$\neq 0$	no	/
$\neq 0$	0	0	yes	$\alpha\vec{v}_1$
$\neq 0$	0	$\neq 0$	yes	$\alpha\vec{v}_1$
$\neq 0$	$\neq 0$	0	no	/
$\neq 0$	$\neq 0$	$\neq 0$	no	/

$\lim_{n\to\infty}(\gamma(\tfrac{1}{2})^n\vec{v}_3) \rightarrow \vec{0}$

$\lim_{n\to\infty}(\beta(2)^n\vec{v}_2) \rightarrow \vec{\infty}$

$\lim_{n\to\infty}(\beta(2)^n\vec{v}_2 + \gamma(\tfrac{1}{2})^n\vec{v}_3) \rightarrow \vec{\infty} + \vec{0}$

$\lim_{n\to\infty}(\alpha(1)^n\vec{v}_1) \rightarrow \alpha\vec{v}_1$

$\lim_{n\to\infty}(\alpha(1)^n\vec{v}_1 + \gamma(\tfrac{1}{2})^n\vec{v}_3) \rightarrow \alpha\vec{v}_1 + \vec{0}$

$\lim_{n\to\infty}(\alpha(1)^n\vec{v}_1 + \beta(2)^n\vec{v}_2) \rightarrow \alpha\vec{v}_1 + \vec{\infty}$

$\lim_{n\to\infty}(\alpha(1)^n\vec{v}_1 + \beta(2)^n\vec{v}_2 + \gamma(\tfrac{1}{2})^n\vec{v}_3) \rightarrow \alpha\vec{v}_1 + \vec{\infty} + \vec{0}$