Dis 5B EECS16A

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1. Diagonalization

One of the most powerful ways to think about matrices is to think of them in diagonal form [1].

(a) Consider following three matrices: a matrix $\mathbf{A}$, a matrix $\mathbf{V}$ whose columns are the eigenvectors of $\mathbf{A}$, and finally a diagonal matrix $\mathbf{\Lambda}$ with the eigenvalues of $\mathbf{A}$ on the diagonal (in the same order as the eigenvectors (or columns) of $\mathbf{V}$). From these definitions, show that

$$\mathbf{A}\,\mathbf{V} = \mathbf{V}\,\mathbf{\Lambda}$$

[1] Not all matrices can be put in this form but most can. The ones that can't be diagonalized can be put in a similar form called the Jordan form.

$$V = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix}$$

* $A\vec{v}_i = \lambda_i \vec{v}_i$

$$AV = A\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$V\Lambda = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_n \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2\vec{v}_2 & \cdots & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

same ✓

$$\begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{21} \\ \lambda_1 v_{12} & \lambda_2 v_{22} \end{bmatrix}$$

$$AV = V\Lambda$$
$$AVV^{-1} = V\Lambda V^{-1}$$
$$A = V\Lambda V^{-1}$$

(b) By multiplying both sides on the right by $\mathbf{V}^{-1}$, we get following expression: $\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}$. This is called the diagonal form of $\mathbf{A}$. Using this expression, we can consider the action of $\mathbf{A}$ on a coordinate vector $\vec{r}^{(u)}$ in the standard basis. Please interpret each step of the following calculation in terms of coordinate transformations and scaling by eigenvalues.

$$\mathbf{A}\vec{r}^{(u)} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^{-1}\vec{r}^{(u)}$$

* recall change of basis:

$$V\vec{r}^{(v)} = U\vec{r}^{(u)} = \vec{r}$$

$$\vec{r}^{(v)} = V^{-1}U\vec{r}^{(u)}$$ changing from u basis to v basis

$$\vec{r}^{(u)} = U^{-1}V\vec{r}^{(v)}$$ from v basis to u basis

* note: $\vec{r}^{(u)}$ is in the standard basis

$$U = I$$

$$\vec{r}^{(v)} = V^{-1}I\vec{r}^{(u)} = V^{-1}\vec{r}^{(u)}$$

$$\vec{r}^{(u)} = I^{-1}V\vec{r}^{(v)} = V\vec{r}^{(v)}$$

$$A\vec{r}^{(u)} = V\Lambda V^{-1}\vec{r}^{(u)}$$

$V^{-1}\vec{r}^{(u)}$: coord of $\vec{r}^{(v)}$ in eigenvector basis

$\Lambda V^{-1}\vec{r}^{(u)}$: scaling coord of $\vec{r}^{(v)}$ by eigenvalues

$V\Lambda V^{-1}\vec{r}^{(u)}$: transforming back to standard basis

2. Fibonacci Sequence

Learning Goal: *This problem uses the famous Fibonacci sequence to show how writing a matrix in terms of its eigenvalues and eigenvactors can be used to analyze the matrix raised to a power.*

(a) The Fibonacci sequence can be constructed according to the following relation. The Nth number in the Fibonacci sequence, F_N, is computed by adding the previous two numbers in the sequence together:

$$F_N = F_{N-1} + F_{N-2}$$

We select the first two numbers in the sequence to be $F_1 = 0$ and $F_2 = 1$ and then we can compute the following numbers as

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Explicitly write down the matrix **A** and its elements that can be used to compute the next value of the Fibonacci sequence, F_N, from the previous two values, F_{N-1} and F_{N-2}.

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix}$$

(b) Construct a matrix **P** whose columns are the eigenvectors of **A**, and a matrix **D** whose diagonal elements are the corresponding eigenvalues of **A**.

P will have the form $\begin{bmatrix} | & | \\ \vec{p}_1 & \vec{p}_2 \\ | & | \end{bmatrix}$ where $\vec{p}_1$ and $\vec{p}_2$ are the first and second eigenvectors of **A**.

D will have the form $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are the first and second eigenvalues of **A**.

Show your work and confirm that $\mathbf{A} = \mathbf{PDP}^{-1}$. (No need to provide a proof).

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = (1-\lambda)(-\lambda) - (1)(1) = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{+1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \left(A - \left(\frac{1+\sqrt{5}}{2}\right)I\right)\vec{p}_1 = \vec{0}$$

← eigenvector

$$* \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right) = \frac{1-5}{4} = \frac{-4}{4} = -1$$

$$\frac{2-1-\sqrt{5}}{2}$$

$$\left[\begin{array}{cc|c} 1-\left(\frac{1+\sqrt{5}}{2}\right) & 1 & 0 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{array}\right] = \left[\begin{array}{cc|c} \frac{1-\sqrt{5}}{2} & 1 & 0 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{array}\right] \xrightarrow{R_1 \leftarrow -\left(\frac{1+\sqrt{5}}{2}\right)R_1} \left[\begin{array}{cc|c} 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 - \left(\frac{1+\sqrt{5}}{2}\right)x_2 = 0$ $\qquad x_1 = \frac{1+\sqrt{5}}{2}\alpha$

$x_2 = \text{anything} = \alpha$ $\qquad x_2 = \alpha$

$$\vec{p}_1 = \alpha \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2} \qquad \vec{p}_2 = \alpha \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

$$* \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$D = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$A = PDP^{-1} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}\right) = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

✓

(c) **(Challenge):** Consider now how to compute $\mathbf{A}^N$ using the result that $\mathbf{A} = \mathbf{PDP}^{-1}$. We can write $\mathbf{A}^N = (\mathbf{PDP}^{-1})^N = \mathbf{PD}^N\mathbf{P}^{-1}$. Can you justify that? Use that equation to confirm that

$$F_N = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{N-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{N-1}$$

is an analytical expression for the Nth Fibonacci number.

$A^N = PD^NP^{-1}$

$\begin{bmatrix} F_3 \\ F_2 \end{bmatrix} = A \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$

$\begin{bmatrix} F_4 \\ F_3 \end{bmatrix} = A \begin{bmatrix} F_3 \\ F_2 \end{bmatrix} = A^2 \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}$

$$\begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = A \begin{bmatrix} F_{N-1} \\ F_{N-2} \end{bmatrix} = A^{N-2} \begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = A^{N-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F_N = [1 \;\; 0] \begin{bmatrix} F_N \\ F_{N-1} \end{bmatrix} = [1 \;\; 0]\, A^{N-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \;\; 0] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{N-2} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

1×2 $\quad 2\times 1$ $\quad F_N \cdot 1 + 0 \cdot F_{N-1}$ $\quad PD^{N-2}P^{-1}$

$(ABC = A(BC))$

$$= [1 \;\; 0] \begin{bmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt5}{2} & 0 \\ 0 & \frac{1-\sqrt5}{2} \end{bmatrix}^{N-2} \left(\frac{1}{\sqrt5} \begin{bmatrix} 1 & \frac{-1+\sqrt5}{2} \\ -1 & \frac{1+\sqrt5}{2} \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt5}{2}\right)^{N-2} & 0 \\ 0 & \left(\frac{1-\sqrt5}{2}\right)^{N-2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\sqrt5}{2} & \frac{1-\sqrt5}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{N-2} \\ \frac{-1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{N-2} \end{bmatrix} = \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{N-1} - \frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{N-1}$$ ✓

3. (PRACTICE) Matrix Powers

One of the most powerful things about matrix diagonalization is that it gives us some insight into polynomial functions of matrices. Suppose we have some diagonalizable matrix $\mathbf{A}^{n \times n}$:

$A = V\Lambda V^{-1}$

(a) Write $\mathbf{A}^N$ using the diagonalization of $\mathbf{A}$, simplify the matrix product as much as possible.

$$\begin{aligned} A^2 &= AA \\ &= (V\Lambda V^{-1})(V\Lambda V^{-1}) \\ &= V\Lambda (V^{-1}V) \Lambda V^{-1} \\ &= V\Lambda I \Lambda V^{-1} \\ &= V\Lambda\Lambda V^{-1} \\ &= V\Lambda^2 V^{-1} \end{aligned}$$

$$\begin{aligned} A^3 &= A^2 A \\ &= V\Lambda^2 V^{-1} V \Lambda V^{-1} \\ &= V\Lambda^3 V^{-1} \end{aligned}$$

$$A^N = V\Lambda^N V^{-1}$$

(b) Given $\mathbf{A}^N$ in the simplified form as found above, explicitly write out the elements of the expression's matrix product of diagonal eigenvalue matrices $\mathbf{\Lambda}$. What happens to the diagonal elements?

$$\Lambda^N = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots 0 & \lambda_n \end{bmatrix}^N = \begin{bmatrix} \lambda_1^N & 0 & \cdots & 0 \\ 0 & \lambda_2^N & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots 0 & \lambda_n^N \end{bmatrix}$$

Ex: $$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

(c) Now that we fully understand the diagonal expansion of $\mathbf{A}^N$, what if I told you that this formula works beyond natural numbers like $N = 1, 2, \ldots$? Use this formula to express the inverse $\mathbf{A}^{-1}$. Does this agree with what we have seen for inverses of matrix products? And what happens if $\lambda_1 = 0$?

$$A^{-1} = (V \Lambda V^{-1})^{-1} = (V^{-1})^{-1} \Lambda^{-1} V^{-1}$$

$$= V \Lambda^{-1} V^{-1}$$

makes sense

$$\Lambda^{-1} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \frac{1}{\lambda_n} \end{bmatrix}$$

$\lambda_1 = 0$? $\quad A^{-1}$ $\quad \frac{1}{\lambda_1} \to \infty$

can't diagonlize

$\lambda = 0$ → nontrivial nullspace

→ not invertible

$$A\vec{v} = \lambda \vec{v}$$

$$A^{-1}A\vec{v} = A^{-1}(\lambda \vec{v})$$

$$\vec{v} = \lambda A^{-1}\vec{v}$$

$$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$$

* A^{-1} has same eigenvectors but now $\lambda \to \frac{1}{\lambda}$

* $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$