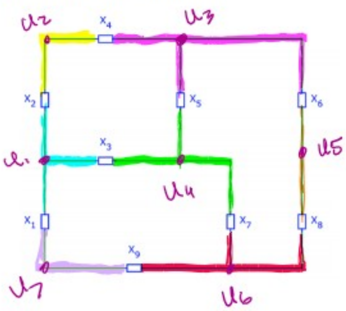
Dis 6A EECS16A

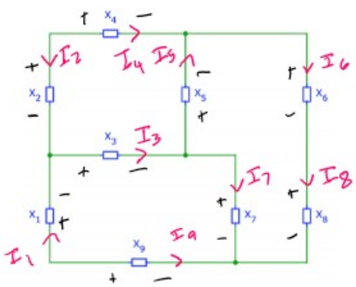
tinyurl.com/miyuki-feedback

1. Label the circuit

In the circuit shown below, label all the nodes, and show one possible way of labeling all the element voltages and currents following the passive sign convention.

7 nodes





circuit element: $(x_1, x_2, \dots, x_8)$

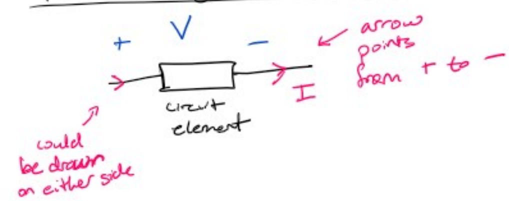
ex: resistors, voltage sources, current sources, capacitors

node: location where 2 or more circuit elements meet

* section of wire/conductive material

* voltage is the same everywhere along a node

passive sign convention



Midterm 1 Review:

We probably won't get to every problem so use the poll to choose which are your top 2 problems that you would like to go over.

2. Solving Systems of Equations

Topics: system of equations, Gaussian elimination, number of solutions.

A system of linear equations can either have one solution, an infinite number of solutions, or no solution at all. Solve the following systems of equations by Gaussian elimination, and state the number of solutions.

(a) Write the following system of equations in augmented matrix form:

$$x + 3y = 4 \quad (1)$$
$$-2x - 5y = -6 \quad (2)$$

(b) Once in augmented matrix form we can use Gaussian Elimination to solve the system of equations. See what solution you get using Gaussian elimination.

(c) Solve the following system of equations:

$$x + 3y - z = 4 \quad (3)$$
$$4x - y + 2z = 8 \quad (4)$$
$$2x - 7y + 4z = -3 \quad (5)$$

(d) Solve the following system of equations:

$$x + 2y = 4 \quad (6)$$
$$2y = 2 \quad (7)$$
$$2x + 4y + z + w = 14 \quad (8)$$
$$z + w = 6 \quad (9)$$

3. Proof

Topics: proof, null space, invertibility.

Consider a square matrix $\mathbf{A}$. Prove that if $\mathbf{A}$ has a non-trivial nullspace, i.e. if the nullspace of $\mathbf{A}$ contains more than just $\vec{0}$, then matrix $\mathbf{A}$ is not invertible.

4. StateRank Car Rentals

Topics: transition matrix, eigenvalue and eigenvector, matrix inversion, steady state.

You are an analyst at StateRank Car Rentals, which operates in California, Oregon, and Nevada. You are hired to analyze the number of rental cars going into and out of each of the three states (CA, OR, and NV).

The number of cars in each state on day $n \in \{0, 1, \dots\}$ can be represented by the state vector $\vec{s}[n] = \begin{bmatrix} s_{CA}[n] \\ s_{OR}[n] \\ s_{NV}[n] \end{bmatrix}$.

The state vector follows the state evolution equation $\vec{s}[n+1] = \mathbf{A}\vec{s}[n], \forall n \in \{0, 1, \dots\}$, where the transition matrix, $\mathbf{A}$, of this linear dynamic system is

$$\mathbf{A} = \begin{bmatrix} [illegible] \end{bmatrix}$$

(a) Use the designated boxes in Figure 1 to **fill in the weights for the daily travel dynamics of rental cars between the three states**, as described by the state transition matrix $\mathbf{A}$.

Note the order of the elements in the state vector $\vec{s}[n]$.

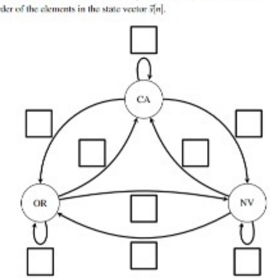


Figure 1: StateRank Rental Cars Daily Travel Dynamics.

(b) Suppose the state vector on day $n = 4$ is $\vec{s}[4] = \begin{bmatrix} 100 \\ 200 \\ 100 \end{bmatrix}$. **Calculate the state vector on day 5, $\vec{s}[5]$.**

(c) We want to express the number of cars in each state on day n as a function of the initial number of cars in each state on day 0. That is, we write $\vec{s}[n]$ in terms of $\vec{s}[0]$ as follows:

$$\vec{s}[n] = \mathbf{B}\vec{s}[0]$$

Express the matrix B in terms of A and n.

(d) We denote the eigenvalue/eigenvector pairs of the matrix $\mathbf{A}$ by

$$\left(\lambda_1 = 1, \vec{u}_1 = \begin{bmatrix} 50 \\ 40 \\ 110 \end{bmatrix}\right), \left(\lambda_2, \vec{u}_2 = \begin{bmatrix} 0 \\ -10 \\ 10 \end{bmatrix}\right), \text{ and } \left(\lambda_3, \vec{u}_3 = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix}\right).$$

Find the eigenvalues λ_2 and λ_3 corresponding to the eigenvectors $\vec{u}_2$ and $\vec{u}_3$, respectively. Note that since $\lambda_1 = 1$ is given, you don't have to calculate it.

(e) For the given dynamics in this problem, **does a matrix C exists such that $\vec{s}[n-1] = \mathbf{C}\vec{s}[n]$, for $n \in \{1, 2, \dots\}$? Justify your answer.**

(f) Suppose that the initial number of rental cars in each state on day 0 is

$$\vec{s}[0] = \begin{bmatrix} 7000 \\ 5000 \\ 8000 \end{bmatrix} = 100\vec{u}_1 - 100\vec{u}_2 - 200\vec{u}_3,$$

where $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$ are the eigenvectors from part (d).

After a very large number of days n, how many rental cars will there be in each state?

That is, i) calculate

$$\vec{s}^* = \lim_{n \to \infty} \vec{s}[n]$$

and ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.

Hint: If you didn't solve part (d), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

5. True or False?

Topics: invertibility, span, vector space, basis, linear dependence.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$.

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

(d) The two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ form a basis for the subspace span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$.

(e) The dimension of the subspace span$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ is 3.

(f) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

2. Solving Systems of Equations

Topics: system of equations, Gaussian elimination, number of solutions.

A system of linear equations can either have one solution, an infinite number of solutions, or no solution at all. Solve the following systems of equations by Gaussian elimination, and state the number of solutions.

(a) Write the following system of equations in augmented matrix form:

$$x + 3y = 4 \tag{1}$$
$$-2x - 5y = -6 \tag{2}$$

* for mechanical problems

→ check your answer if you can

$A\vec{x} = \vec{b}$ GE find $\vec{x}$ unique/inf

↳ plug back in

$A\vec{x} = \vec{b}$

inverse of A → find A^{-1}

↳ $A^{-1}A = I$

nullspace A → find $\vec{x}$

↳ $A\vec{x} = \vec{0}$

e-vals/vec → find $\vec{v}, \lambda$

↳ $A\vec{v} = \lambda\vec{v}$

(b) Once in augmented matrix form we can use Gaussian Elimination to solve the system of equations. See what solution you get using Gaussian elimination.

(c) Solve the following system of equations:

$$x + 3y - z = 4 \tag{3}$$
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$$z + w = 6 \tag{9}$$

3. Proof

Topics: proof, null space, invertibility.

Consider a square matrix $\mathbf{A}$. Prove that if $\mathbf{A}$ has a non-trivial nullspace, i.e. if the nullspace of $\mathbf{A}$ contains more than just $\vec{0}$, then matrix $\mathbf{A}$ is not invertible.

Know: A is square

A has a nontrivial nullspace: $A\vec{x} = \vec{0}$ for some $\vec{x} \neq \vec{0}$

show: A is not invertible

↳ $A^{-1}A = I$

2 options

① Proof by construction

thinking: A has a nontrivial nullspace

↳ lin dep cols

↳ A is not invertible

* we have proved

A is invertible ⇒ cols are lin ind

but we don't have the tools to do this yet

② Proof by contradiction

* assume what you want to show is the opposite

→ demonstrate it contradicts what you know

assume A is invertible

$A\vec{x} = \vec{0}$

$A^{-1}A\vec{x} = A^{-1}\vec{0}$

$I\vec{x} = \vec{0}$

$\vec{x} = \vec{0}$

$I\vec{v} = \vec{v}$

$IA = A$

contradicts $\vec{x} \neq \vec{0}$ for nontrivial nullspace

→ therefore A can't be invertible

Strategies for Proofs

- write out mathematical defn for what you know & what you want to show
- try simple examples to find patterns
- manipulate the defns to get from what you know to what you want to show

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$$\mathbf{A} = \begin{bmatrix} \frac{7}{10} & \frac{1}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{6}{10} & \frac{1}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{8}{10} \end{bmatrix}$$

(a) Use the designated boxes in Figure 1 to **fill in the weights for the daily travel dynamics of rental cars between the three states**, as described by the state transition matrix $\mathbf{A}$.
Note the order of the elements in the state vector $\vec{s}[n]$.

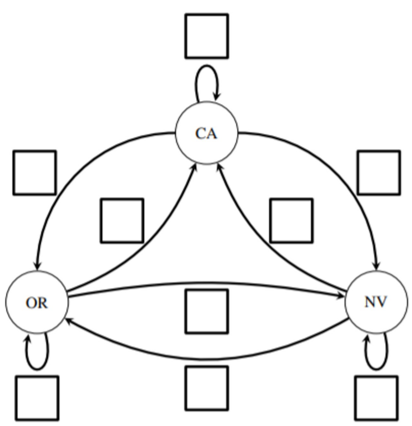

Figure 1: StateRank Rental Cars Daily Travel Dynamics.

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That is, i) calculate

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and ii) show that the system will indeed converge to $\vec{s}^*$ as $n \to \infty$ if it starts from $\vec{s}[0]$.
Hint: If you didn't solve part (d), the eigenvalues satisfy $\lambda_1 = 1, |\lambda_2| < 1$ and $|\lambda_3| < 1$.

5. True or False?

Topics: invertibility, span, vector space, basis, linear dependence.

(a) There exists an invertible $n \times n$ matrix A for which $A^2 = 0$. ← matrix

False

assume A is invertible → $A^{-1}A = I$

$A^2 = 0$

$AA = 0$

$A^{-1}AA = A^{-1}0$ → $IA = 0$

$A = 0$

$AA^{-1} = 0A^{-1}$

$I \neq 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) If A is an invertible $n \times n$ matrix, then for all vectors $\vec{b} \in \mathbb{R}^n$, the system $A\vec{x} = \vec{b}$ has a unique solution.

True

$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$ ← unique

(c) If A and B are invertible $n \times n$ matrices, then the product AB is invertible.

True

know: $A^{-1}A = I \quad B^{-1}B = I$

show: $(AB)^{-1}(AB) = I$

$B^{-1}A^{-1}AB = B^{-1}IB$

$= B^{-1}B$

$= I$ ✓

* $(AB)^{-1} = B^{-1}A^{-1}$

(d) The two vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ form a basis for the subspace $\text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$.

True

→ lin ind

→ span the same space

Ex: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

* not a basis for the same subspace

$$\text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$\vec{v} \in \alpha_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v} \in \alpha_2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \beta_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(e) The dimension of the subspace $\text{span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ is 3. $\in \mathbb{R}^3$

False

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$ dim: 2

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in \mathbb{R}^2$ # of basis vectors

i.e. lin ind vectors that define the span

* need ~~at least~~ n lin ind vecs to span $\mathbb{R}^n$

(f) A set of n linearly dependent vectors in $\mathbb{R}^n$ can span $\mathbb{R}^n$.

False

lin dep vecs

$A = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

$\dim(\text{col}(A)) < n$

can't span $\mathbb{R}^n$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \alpha_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & \cancel{2} \\ 0 & 1 & \cancel{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\} \neq \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$\underbrace{\text{span}\overbrace{\left\{\underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{basis}}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}}^{\text{not a basis}} = \mathbb{R}^3}_{\text{dim} = 3}$$