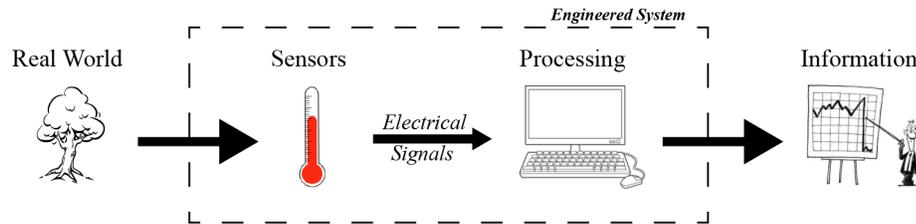


EECS 16A Designing Information Devices and Systems I

Spring 2021 Lecture Notes

Note 11

11.1 Introduction to Electrical Circuit Analysis



Our ultimate goal is to design systems that solve people’s problems. To do so, it’s critical to understand how we go from real-world events all the way to useful information that the system might then act upon. The most common way an engineered system interfaces with the real world is by using sensors and/or actuators that are often composed of electronic circuits; these communicate via electrical signals with processing units, which are *also* composed entirely of electronic circuits. In order to fully understand and *design* a useful system, we will need to first understand **Electrical Circuit Analysis**.

In this note, we will start by intentionally ignoring the underlying physics of electrical circuits, and will instead focus on a standard procedure and set of rules that will allow us to systemically “solve” such circuits, (i.e., given a circuit diagram, solve for all of the relevant electrical quantities in that circuit). By abstracting the physics away during the analysis step, we hope to emphasize that the analysis in and of itself is simply a matter of taking a visual diagram (representing an electrical circuit) and applying a set of rules to it that will convert the diagram into a set of (linear) equations that can then be solved using the techniques we have developed in the first module.

11.2 Basic Circuit Quantities

Let’s start with some definitions of basic quantities present in an electrical circuit. **Current** is the flow of charges (e.g. electrons) in the circuit, and **voltage** is the potential energy (per charge) between two points in the circuit. This potential energy is what causes charge to flow (ie. causes current). **Resistance** is the material’s tendency to resist the flow of current.

Quantity	Symbol	Units
Voltage	V	Volts (V)
Current	I	Amperes (A)
Resistance	R	Ohms (Ω)

We use these quantities in a **Circuit Diagram**, a visual representation of how a collection of circuit elements are connected. Each circuit element has some voltage *across* it and some current *through* it. We often refer to the voltage difference across some element as the *voltage drop* across the element.

Why is voltage “across” a circuit element? Voltage, or electric potential, is only defined *relative* to another point. A simple analogy is elevation: A mountain’s summit could be 9,000 ft above sea level, but 21,000 ft above the bottom of the ocean. In both cases, the elevation is only meaningful relative to another point. For convenience, we frequently define sea level as a reference point with “0 ft of elevation” – then we can state elevation as a single number which is implicitly referenced to sea level (ex. the mountain is 9,000 ft tall). Similarly, in circuits, we will frequently define a reference point, called **ground**, against which other voltages can be measured.

11.3 Basic Circuit Elements

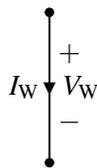
How do our basic circuit quantities interact? It depends on the circuit element! For each element, we will define a relationship between the voltage across the element and the current through it. We will call this the “IV Relationship”, because current is represented by I and voltage by V. By convention, for these elements, we will define some positive “+” terminal and some negative “-” terminal, and we define the voltage across the element to be the difference between the positive terminal and the negative terminal. Let’s look at some of the most common circuit elements and their IV relationships.

Wire: The most common element is the wire. A wire connects two circuit elements, and we assume current flows freely through it, without any voltage drop. In a circuit diagram, any solid line (that is not a circuit element) is assumed to be a wire. The IV relationship for a wire is:

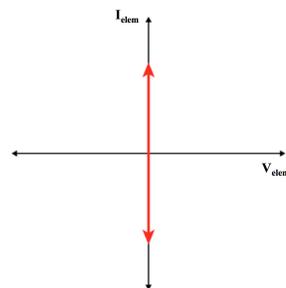
$V_W = 0$ A wire is an ideal connection with zero voltage across it.

$I_W = ?$ The current through a wire can take any value, and is determined by the rest of the circuit.

Symbol



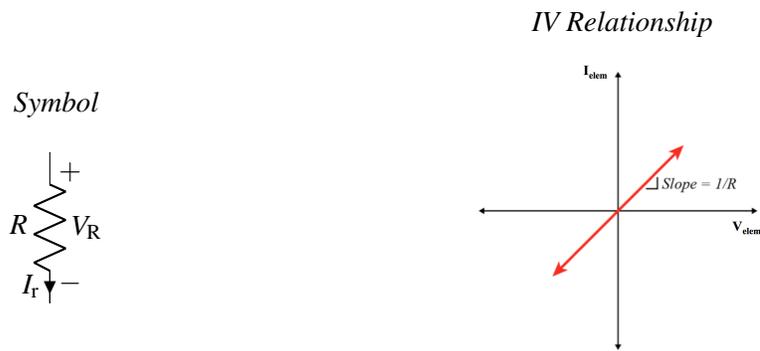
IV Relationship



Resistor: A resistor is an element that restricts or “resists” current, which is the flow of charge. This resistance causes a voltage difference across the resistor. The IV relationship of a resistor is called “Ohm’s Law.”

$V_R = I_R R$ The voltage across a resistor is determined by Ohm’s Law.

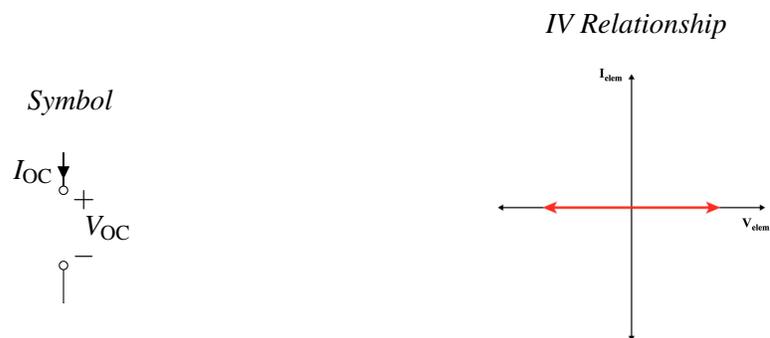
$I_R = \frac{V_R}{R}$ The current through a resistor is determined by Ohm’s Law.



Open Circuit: Whereas a wire freely allows current flow with no voltage drop, an open circuit does not allow any current flow through it. As a result, it can have any voltage across it. Physically, an open circuit exists between any two points that are not connected, such as a disconnected wire.

$V_{OC} = ?$ The voltage across an open circuit can take any value, and is determined by the rest of the circuit.

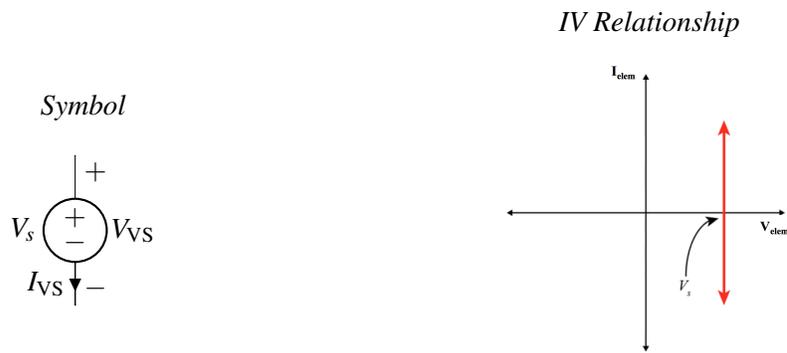
$I_{OC} = 0$ No current is allowed to flow through an open circuit.



Voltage Source: A voltage source is a component that forces a specific voltage V_s across its terminals. The most common voltage sources we encounter are batteries, but these are not very precise in general. In the lab, we may use specific tools to give us very precise voltages. The + and - sign indicates which direction the voltage is pointing. The voltage difference between the “+” terminal and the “-” terminal is always equal to V_s , no matter what else is happening in the circuit. A voltage source is expected to allow any current through it.

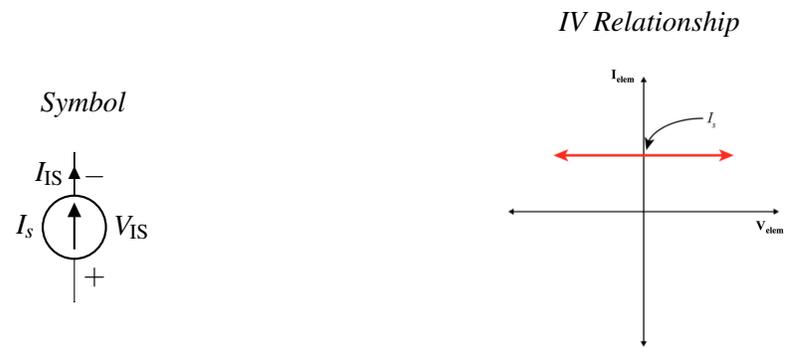
$V_{VS} = V_s$ The voltage across the voltage source is always equal to the source value.

$I_{VS} = ?$ The current through a voltage source is determined by the rest of the circuit.

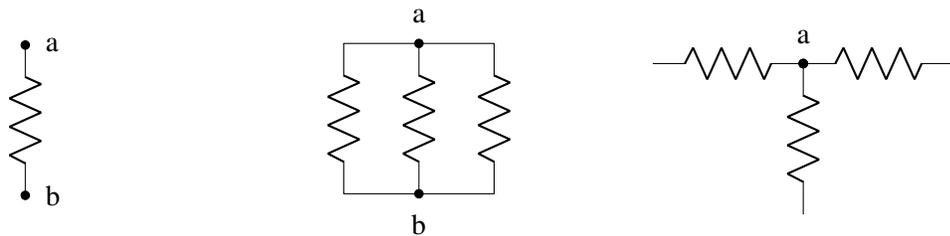


Current Source: A current source forces a specific current I_s in the direction specified by the arrow indicated on the schematic symbol. The current flowing through a current source is always equal to I_s , no matter what else is happening in the circuit. A current source is expected to allow any voltage across it. Note the duality between this element and the voltage source.

- $V_{IS} = ?$ The voltage across a current source is determined by the rest of the circuit.
- $I_s = I_s$ The current through a current source is always equal to the source value.



Node: A node is a place in a circuit where two or more circuit elements meet. Everything at a given node shares the same voltage potential, which we may also call the *node potential*. Nodes are not necessarily just defined as a single point, but by everything between the circuit elements. Wires are considered parts of nodes because they have 0 voltage drop and so have the same potential as everything it connects. We will take a deeper dive at identifying nodes later in this lecture note.



Here are 3 examples of nodes. Nodes are marked by black dots and labeled, in this case a and b . Note that while the black dot denotes where the node is, the actual node extends all the way to the resistor terminals.

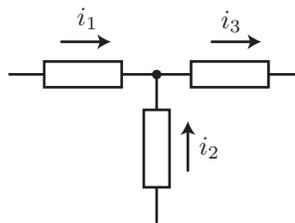
Branches: A branch in a circuit is any connection between two nodes. Generally, this is a single 2-terminal element, such as a resistor or a voltage source. We won't use this term frequently in this class, but you may see it in other literature. A term frequently associated with branches is *branch current*, which is the current through a given branch. This is simply equal to the current through the branch's associated circuit element.

11.4 Rules for Circuit Analysis

In addition to the IV relationships for a single elements, there are also rules govern the current and voltage relationships between multiple elements.

11.4.1 Kirchhoff's Current Law (KCL)

Kirchhoff's Current Law (KCL) states that the net current flowing out of (or equivalently, into) any node of a circuit is zero. To put this more simply, the current flowing into a node must equal the current flowing out of that node.

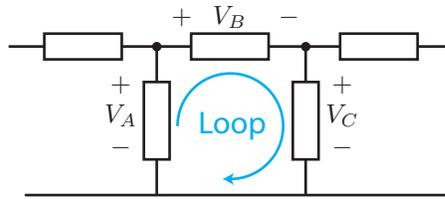


For example, consider the circuit shown above. We define current flowing out of the node to be positive, and therefore, current flowing into the node is negative. For example, from the left branch, there is i_1 current flowing into the node. We can equivalently say $-i_1$ current flowing out of the node. Similarly, there is i_2 current flowing into the node from the bottom branch and i_3 current flowing out of the node through the right branch. In total, we have $i_1 + i_2$ flowing into the node, and i_3 flowing out of the node. Therefore, the currents must satisfy

$$(-i_1) + (-i_2) + i_3 = 0 \text{ or } i_1 + i_2 = i_3 \quad (1)$$

11.4.2 Kirchhoff's Voltage Law (KVL)

Kirchhoff's Voltage Law (KVL) states that the sum of voltages across the elements connected in a loop must be equal to zero. The illustration below shows an example of a loop in a circuit. In our elevation analogy for voltage, this is equivalent to saying "what goes up must come down". We will not actively use KVL in the analysis procedure we outline next, but it is still an important tool and can be used as an additional rule to double check your final answers.



Mathematically, KVL states that:

$$\sum_{\text{Loop}} V_k = 0. \quad (2)$$

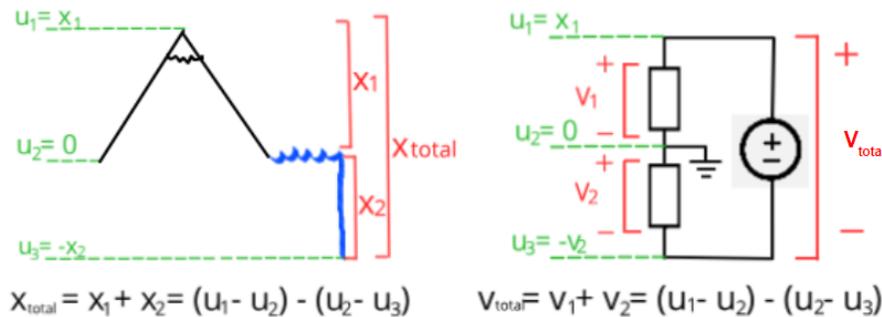
When adding the voltage drops around the loop, we must follow a convention. If the arrow corresponding to the loop goes into the “-” terminal of an element and out the “+” terminal, we add the voltage across that element. (In our elevation analogy, we went “uphill” from a lower voltage to a higher voltage so we gained “elevation.”) Conversely, if the arrow goes into the “+” terminal of an element and out of the “-” terminal, we subtract the voltage across that element (this is like going “downhill” in our elevation analogy). Following this convention for the example in above illustration we find:

$$V_A - V_B - V_C = 0. \quad (3)$$

Note that if we had defined the loop in the opposite direction, we would find this instead:

$$-V_A + V_C + V_B = 0. \quad (4)$$

Thinking about an elevation analogy to voltage can help give some intuition to KVL: If you walk in a circle (a loop) so that you end up back where you started, then your total change in elevation must be zero, no matter how much you go up or down. If you walk in a line, ending up somewhere different, then your total change in elevation is equal to the sum of all of the elevation changes along the way.



11.4.3 Ohm's Law and Resistors

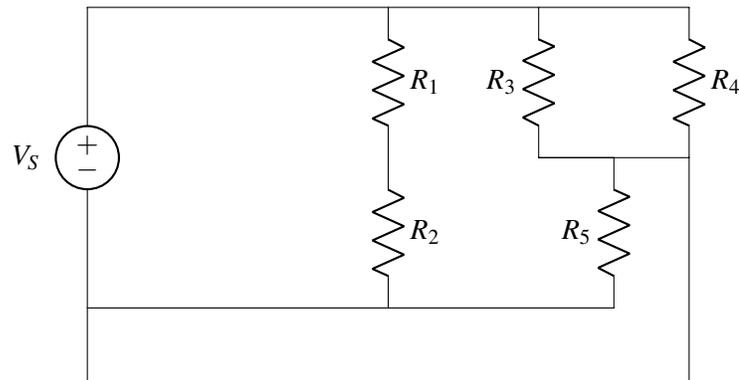
As already described when we introduced resistors as an element, for these elements, the voltage across them is directly proportional to the current that flows through them, where the proportionality constant is the “resistance” (R) of the device. This relationship is known as **Ohm's Law**.

$$V_R = I_R R. \quad (5)$$

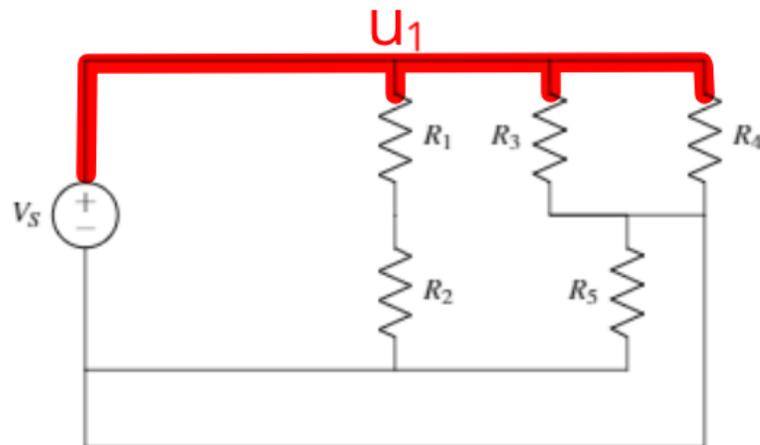
The unit of R is Volts/Amperes, or more commonly “Ohms” (Ω).

11.5 Guide to Finding Nodes

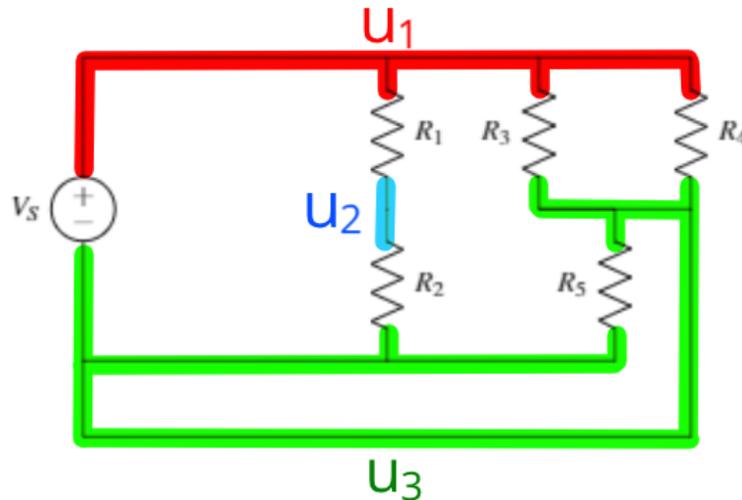
Solving a circuit means finding all the node voltages and branch currents. How do we know which nodes we care about? Here, we will go over a method you can use to correctly identify all of the nodes in a circuit. We’ll go through this method while applying it to an example circuit, shown below.



We’ll find nodes one at a time with this method (the order we find the nodes is arbitrary). We start by choosing a color to represent the first node (red, in this example). Then we choose a starting point on the circuit, such as the upper left corner. From this point, we trace (in red) along all of the connected wires until we hit a non-wire component. Everything traced in red is part of a single node.



Then we choose a new color and a new point on the circuit *that is not already colored*. We repeat the process: tracing over all wires that are connected to this new point and stop when a non-wire element is reached. Repeat this process for new colors until all of the wires have been colored.



We've now identified all of the nodes! There are three in this example circuit. Note that one of these would be labeled "ground" in the seven step circuit analysis procedure, so we would only have to solve for two unknown node potentials.

11.5.1 Redrawing Circuit Diagrams

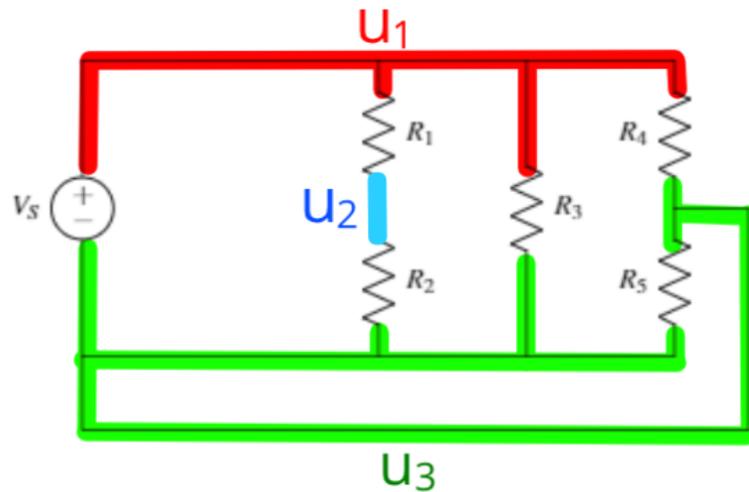
Once we've identified the nodes, it may be useful to redraw the circuit. Redrawing a circuit means drawing a new diagram with the same circuit elements, potentially rearranged, but connected such that the currents and voltages in each element is the same as the original circuit. This new diagram will represent the same circuit behavior as long as the elements are connected the same as before.

The most basic reason to do this is to make a diagram more organized and easier to read. Large circuits can quickly become messy and complicated, so redrawing can help us quickly reorganize. Keeping your circuit elements aligned and orderly, your wire connections short and straight, etc. will help you communicate your circuits clearly.

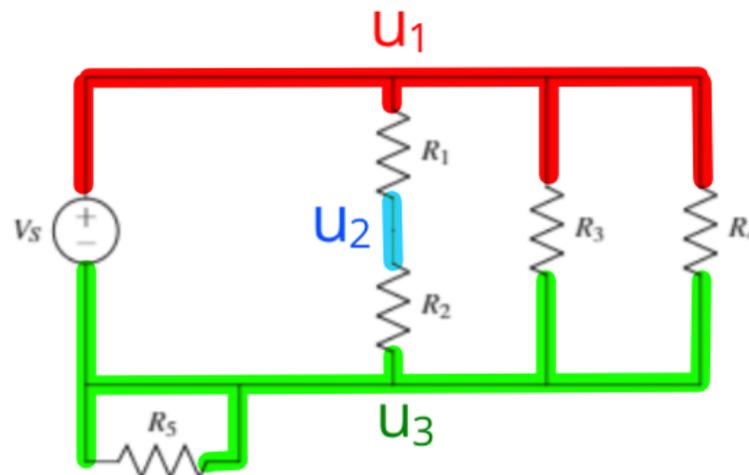
Often though, we will redraw circuit diagrams to look like one of many common classes of circuits, such as a resistive voltage divider. Redrawing complicated circuits to look like these common circuits is very useful because it helps us to see patterns in a circuit diagram, making them easier to analyze and help us build intuition. We will begin to explore these circuit classes in the coming weeks.

It is hugely important when we redraw a circuit that we do not change the circuit behavior. If we don't want the circuit to change when we redraw it, **each terminal of each element must stay connected to its original node**. This is because the voltage drop over each element is dependent only on the nodes it is connected to, and the current through each element is determined by the voltage drop and the IV relationship of that element.

Let's try redrawing our example circuit. We can first move R_3 to a different location. Since R_2 and R_3 share a node (u_3), it is useful to draw them in a way that makes it easy to see that. As long as one end of R_3 is still connected to the red node (u_1) and the other end of R_3 is still connected to the green node (u_3), the circuits are equivalent.



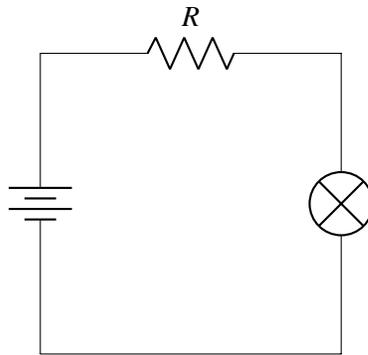
For similar reasons, we may want to do the same to R_4 . Note that R_5 is connected to the green node (u_3) on both sides. Since it does not interact with the red node (u_1), it is useful to move it "out of the way" so we can focus on the R_3 and R_4 :



Note that all of the elements have the same node connections as they did in the original circuit. Therefore this circuit will have the exact same behavior as the original.

11.6 Solving Circuits

In this course, we will learn how to take a real world system and build a circuit diagram that models the behavior of that system, and we will design our own circuits for specific real world tasks. Given a circuit diagram, we want to be able to solve for all the node voltages and element currents. This analysis gives us a complete characterization and understanding of the circuit and can help us solve all sorts of problems.



For example, consider this simple light bulb circuit. This circuit consists of a battery, a resistor, and a lightbulb (represented by a circle with an X). In this circuit, there is some current flowing through the battery, resistor, and light bulb; and there is also some voltage drop across the resistor and light bulb. Solving the circuit will give us these quantities. The current and voltage will give us information about how bright the bulb is, how much power is being dissipated, etc. We can use this information to further design the system for safe amount of current or for a certain brightness.

While this example may seem simple, the same principles can be used to design touchscreens, biosensors, power grids, etc. All of these boil down to deriving an appropriate circuit model and solving it!

11.7 Node Voltage Analysis (NVA)

Node Voltage Analysis, also known as Nodal Analysis, is a powerful systematic tool for finding all the voltages and currents in an electronic circuit. It can be used to solve any circuit composed of linear components, e.g. voltage sources, current sources, and resistors¹. Because the equations we use are all linear, we can formulate the problem using matrices and solve them with the same matrix tools we have been using up till now. In fact, one of the earliest circuit solvers SPICE (Simulation Program with Integrated Circuit Emphasis), which was first published here at Berkeley in 1973, used NVA as its backbone, using efficient matrix algorithms to solve general circuit problems.

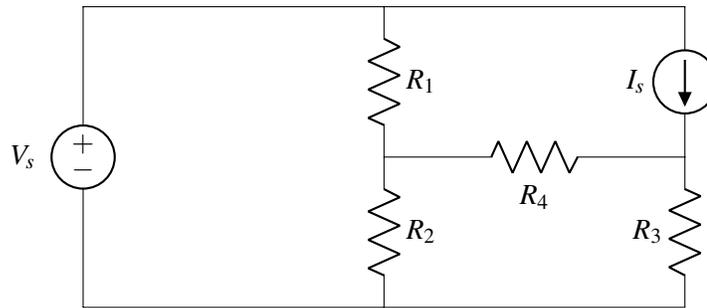
11.7.1 Objective

Find all voltages (and currents) in an electronic circuit.

11.7.2 Procedure

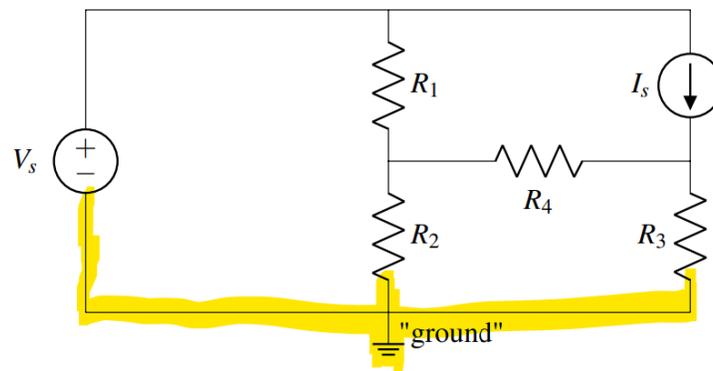
The method proceeds in several steps, illustrated below for the following example circuit:

¹We will see in EECS16B that in certain contexts, we can also use NVA to solve circuits with inductors and capacitors.



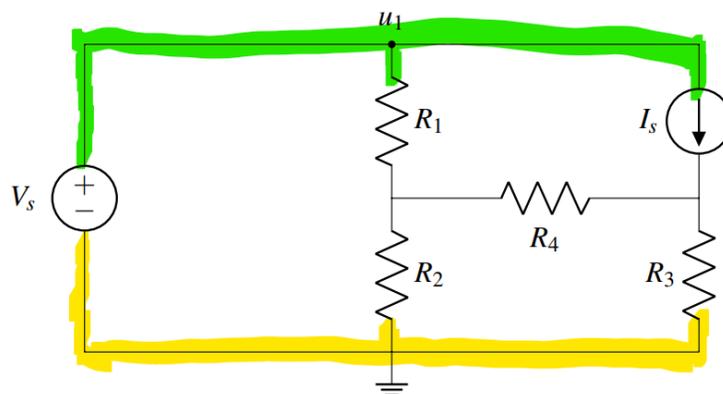
- **Step 1:** Reference Node

Select a reference (ground) node. Any node can be chosen for this purpose. In this example, we choose the node at the bottom of the circuit diagram and draw the "ground" symbol on it.



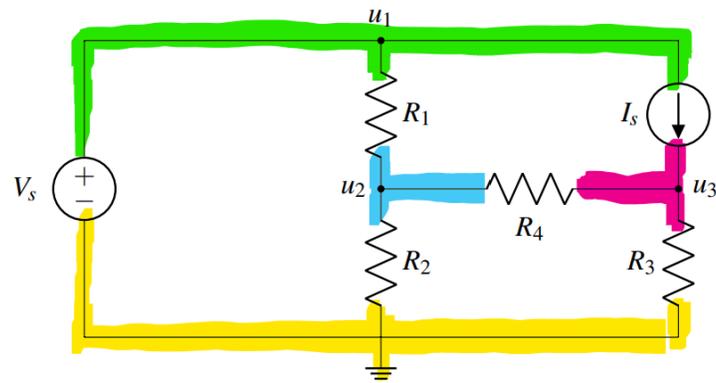
- **Step 2:** Label Nodes with Voltage set by Voltage Sources

Voltage sources like batteries set the voltage of the node they are connected to. In the example, there is only one source, V_s , and we label the corresponding node u_1 (names are arbitrary, but must be unique).



- **Step 3:** Label Remaining Nodes

Now we label all remaining nodes in the circuit except the reference. In this example there are two, u_2 and u_3 .

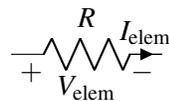


- **Step 4:** Label Element Voltages and Currents

Next we mark the voltage across each element and the currents through each element with unique variables.

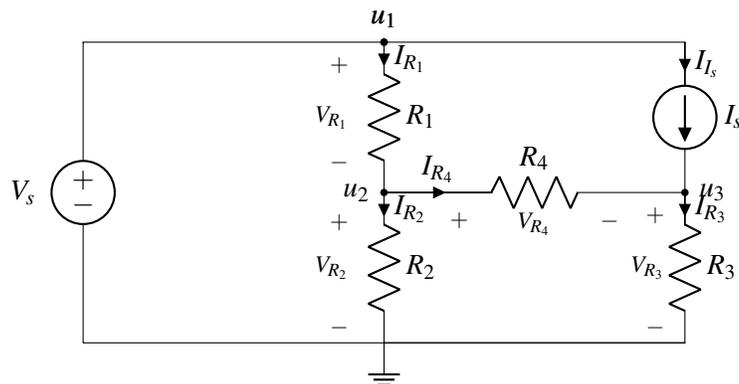
Start with labelling the current through each element. The direction is arbitrary (top to bottom, bottom to top, it won't matter, but stick with your choice in subsequent steps).

Next, mark the element voltages following the passive sign convention. The **passive sign convention** dictates that positive current should *enter* the positive voltage terminal and *exit* the negative voltage terminal of a passive element. This gives us a consistent way to associated the direction of the current and the voltage drop. Below is an example for a resistor:



When we discuss *power* later in the module, you will see why we call this convention “passive.”

The element voltage for I_s is not marked in the example since it will not be needed in the calculations below. Same for the voltage source. There is no harm in marking those, too.



- **Step 5:** Setup equations with Kirchoff's Current Law (KCL)

Write KCL equations for all nodes with unknown voltage, u_2 and u_3 in the example.

At u_2 we get the sum of all currents entering the node equals the sum of all currents exiting:

$$I_{R_1} = I_{R_2} + I_{R_4}$$

Similarly for u_3 :

$$I_{R_4} + I_{I_s} = I_{R_3}$$

- **Step 6: Element Currents**

Find expressions for all element currents in terms of the element voltages and IV characteristics (e.g. Ohm's law). In this step, we will skip voltage sources, since their currents are not defined by IV characteristics but instead by other parts of the circuit.

In this example, there are five elements that are not voltage sources: R_1 , R_2 , R_3 , R_4 , and I_s . We can define the element current for each of these using their IV characteristic (e.g. Ohm's Law):

$$\begin{aligned} I_{R_1} &= \frac{V_{R_1}}{R_1} \\ I_{R_2} &= \frac{V_{R_2}}{R_2} \\ I_{R_3} &= \frac{V_{R_3}}{R_3} \\ I_{R_4} &= \frac{V_{R_4}}{R_4} \\ I_{I_s} &= I_s \end{aligned}$$

- **Step 7: Element Voltages**

Substitute all element voltages in your step 6 equations with differences between node voltages. For example, $V_{R_1} = u_1 - u_2$ and $V_{R_2} = u_2 - 0 = u_2$. At this step, we will also process the voltage sources and look at the nodes they define. For example, the source V_s sits between ground and u_1 . Therefore, the node voltage at u_1 is $u_1 = 0 + V_s = V_s$. Substituting all our element voltages into the current equations, we get:

$$\begin{aligned} I_{R_1} &= \frac{V_s - u_2}{R_1} \\ I_{R_2} &= \frac{u_2}{R_2} \\ I_{R_3} &= \frac{u_3}{R_3} \\ I_{R_4} &= \frac{u_2 - u_3}{R_4} \\ I_{I_s} &= I_s \end{aligned}$$

- **Step 8: Substitute Element Currents in KCL Equations**

Now we substitute the expressions derived in Step 7 into the KCL equations from Step 5.

$$\frac{V_s - u_2}{R_1} = \frac{u_2}{R_2} + \frac{u_2 - u_3}{R_4}$$

$$I_s + \frac{u_2 - u_3}{R_4} = \frac{u_3}{R_3}$$

Let's reorganize this by grouping the terms with unknowns (u_2 and u_3) on the left side and the terms with only known values on the right:

$$u_2\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) + u_3\left(-\frac{1}{R_4}\right) = \frac{V_s}{R_1}$$

$$u_2\left(-\frac{1}{R_4}\right) + u_3\left(\frac{1}{R_3} + \frac{1}{R_4}\right) = I_s$$

We have arrived at a system of linear equations, with u_2 and u_3 as our variables.

- **Step 9:** Solve

Now we can solve for the unknown node voltages, u_2 and u_3 in the example. First, rearrange terms to cast the equations into our familiar linear algebra form $A\vec{x} = b$:

$$\begin{bmatrix} \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4} & -\frac{1}{R_4} \\ -\frac{1}{R_4} & \frac{1}{R_3} + \frac{1}{R_4} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{V_s}{R_1} \\ I_s \end{bmatrix}$$

Then compute the solution using Gaussian Elimination (or let the computer do the work, here using sympy):

```

from sympy import *
init_printing(use_unicode=True)

R1, R2, R3, R4 = symbols('R1 R2 R3 R4')
Y = Matrix([[ 1/R1+1/R2+1/R4, -1/R4], [-1/R4, 1/R3+1/R4]])

V1, I1 = symbols('V1 I1')
b = Matrix([ V1/R1, I1 ])

Vn1, Vn2 = linsolve((Y, b)).args[0]

```

Algebraic result:

```

>>> Vn1
      R2·(I1·R1·R3 + R3·V1 + R4·V1)
-----
R1·R2 + R1·R3 + R1·R4 + R2·R3 + R2·R4
>>> Vn2
      R3·R4·(I1·(R1·R2 + R1·R4 + R2·R4) + R2·V1)
-----
-R1·R2·R3 + (R3 + R4)·(R1·R2 + R1·R4 + R2·R4)

```

Numerical result:

```

>>> values = {R1:1, R2:2, R3:3, R4:4, I1:0.5, V1:1}
>>>
>>> f"Vn1 = {Vn1.evalf(3, subs=values)} V"
Vn1 = 0.739 V
>>> f"Vn2 = {Vn2.evalf(3, subs=values)} V"
Vn2 = 1.17 V

```

We have now solved for the node voltages u_2 and u_3 , corresponding to $Vn1$ and $Vn2$ in the code respectively.

11.7.3 Branch Currents

Now that we have solved for the node voltages, we can easily solve for any element current or branch current. To solve for a branch current, we can plug in the node voltages we found into the element voltage equations from Step 7. For example, the current I_{R_4} through resistor R_4 is

$$I_{R_4} = \frac{V_{R_4}}{R_4} = \frac{u_2 - u_3}{R_4}$$

Using the node voltages we found for u_2 and u_3 , we get numerically: $I_{R_4} = -0.109$ A.

11.8 References

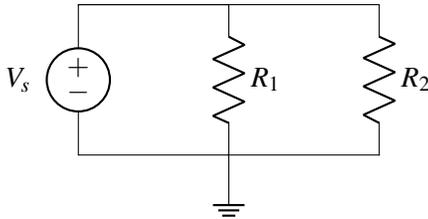
Schaum's Outline of Electric Circuits, Seventh Edition, Section 4.4.

Electronics Reader, Ali Niknejad, 2015, Sections 2.1 and 5.1. (available on course website.)

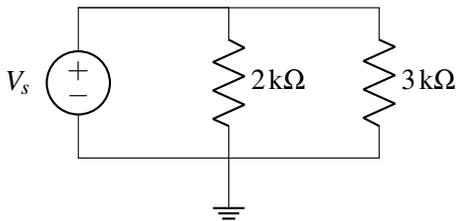
11.9 Practice Problems

These practice problems are also available in an interactive form on the course website (<https://eecs16a.org/#practiceSets>)

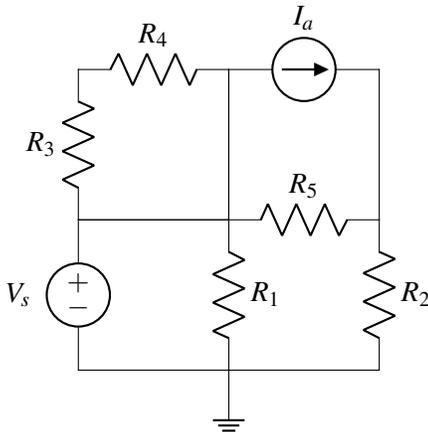
1. True or False: A voltage source can have any current through it.
2. True or False: A current source can have any voltage across it.
3. True or False: The voltage across R_1 and across R_2 is the same.



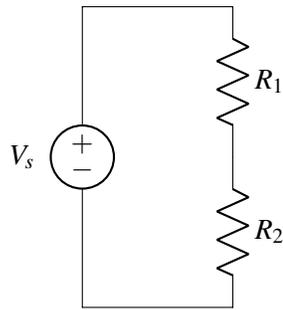
4. True or False: The current through the resistors is the same.



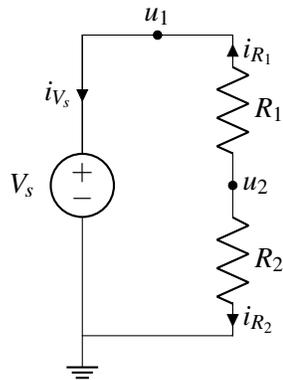
5. If you have n nodes in a circuit with k non-wire elements connecting the nodes, how many equations do you need to solve for all node potentials and element currents? Remember that one node needs to be grounded.
6. How many nodes would you need to label to perform nodal analysis? Include nodes for ground and for V_s .



7. How many nodes are in the following circuit?



8. Assume that you have picked the ground node and labeled the node potentials and branch currents as follows.



What are the $+/-$ labels for R_1 and R_2 according to passive sign convention?

9. For the same circuit as above, formulate a system of equations to solve for all node potentials and branch currents.

