Reference Definitions

Vector spaces: A vector space V is a set of elements that is closed under vector addition and scalar multiplication. For V to be a vector space, the following conditions must hold for every $\vec{u}, \vec{v}, \vec{z} \in V$ and for every $c, d \in \mathbb{R}$:

No escape property (addition) $\vec{u} + \vec{v} \in V$,

No escape property (scalar multiplication) $c\vec{u} \in V$,

Commutativity $\vec{u} + \vec{v} = \vec{v} + \vec{u}$,

Associativity of vector addition $(\vec{u} + \vec{v}) + \vec{z} = \vec{u} + (\vec{v} + \vec{z})$,

Additive identity There is $\vec{0} \in V$, such that $\vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$,

Existence of inverse There is an element $-\vec{u}$, such that $\vec{u} + (-\vec{u}) = \vec{0}$,

Associativity of scalar multiplication $c(d\vec{u}) = (cd)\vec{u}$,

Distributivity of scalar sums $(c + d)\vec{u} = c\vec{u} + d\vec{u}$,

Distributivity of vector sums $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$,

Scalar multiplication identity There is $1\vec{u} = \vec{u}$.

The most important properties are the *no escape properties*. These demonstrate that the vector space is closed under addition and scalar multiplication. That is, if you add two vectors in V, your resulting vector will still be in V. If you multiply a vector in V by a scalar, your resulting vector will still be in V.

Subspaces: A subset W of a *vector space* V is a *subspace* of V if the following three conditions hold for any two vectors $\vec{u}, \vec{v} \in W$ and any scalar $c \in \mathbb{R}$:

Existence of the zero vector $\vec{0} \in W$

No escape property (addition) $\vec{u} + \vec{v} \in W$

No escape property (scalar multiplication) $c\vec{u} \in W$

Note that these are the only properties we need to establish to show that a subset of a vector space is a subspace! The other properties of the underlying vector space come for free, so to speak.

The vector spaces we will work with most commonly are $\mathbb{R}^n$ and $\mathbb{C}^n$ as well as their subspaces.

Basis: A *basis* for a vector space is an *ordered set of linearly independent vectors* that *spans the vector space*.

Therefore, if we want to check whether a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ forms a basis for a vector space V, we check for two important properties:

(a) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

(b) $\text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\} = V$

As we move along, we'll learn how to identify and/or construct a basis, and we'll also learn some interesting properties of bases.

Dimension: The *dimension* of a vector space is the *minimum number* of vectors needed to span the entire vector space. That is, the dimension of a vector space equals the number of vectors in a basis for this vector space.

1. Identifying a Subspace: Proof

Is the set

$$V = \left\{ \vec{v} \;\middle|\; \vec{v} = c\begin{bmatrix}1\\1\\1\end{bmatrix} + d\begin{bmatrix}1\\0\\1\end{bmatrix}, \text{ where } c, d \in \mathbb{R} \right\}$$

a subspace of $\mathbb{R}^3$? Why/why not?

2. Identifying a Basis

Does each of these sets of vectors describe a basis for some subspace?

$$V_1 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$$

3. Exploring Dimension, Linear Independence, and Basis

In this problem, we are going to talk about the connections between several concepts we have learned about in linear algebra – linear independence, dimension of a vector space/subspace, and basis.

Let's consider the vector space $\mathbb{R}^m$ and a set of n vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$.

(a) For the first part of the problem, let $m > n$. Can $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(b) Let $m = n$. Can $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for $\mathbb{R}^m$? Why/why not? What conditions would we need?

(c) Now, let $m < n$. Can $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ form a basis for $\mathbb{R}^m$? What vector space could they form a basis for? (*Hint: Think about whether the vectors can be linearly independent.*)

4. Constructing a Basis

Let's consider a subspace of $\mathbb{R}^3$, V, that has the following property: for every vector in V, the first entry is equal to two times the sum of the second and third entries. That is, if $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \in V$, $a_1 = 2(a_2 + a_3)$.

Find a basis for V. What is the dimension of V?

5. Polynomials as a Vector Space

Let $\mathbb{P}_2$ be the set of polynomials of degree of at most two (that is, $p(t) = at^2 + bt + c$).

(a) Give a basis for $\mathbb{P}_2$.

(b) Consider the linear transformations

$$T_1(f(t)) = 2f(t)$$
$$T_2(f(t)) = f'(t)$$

For each, find the transformation matrix with respect to the basis from part (a).

(c) Suppose that $\{x_0, x_1, x_2\}$ form a basis for $\mathbb{P}_2$ and that the following polynomials have the corresponding coordinates in this basis.

$$(1,1,1) \Rightarrow 2t^2 + 3t$$
$$(1,0,-1) \Rightarrow t + 1$$
$$(0,2,0) \Rightarrow 4t + 2$$

Find the basis vectors x_0, x_1, x_2.