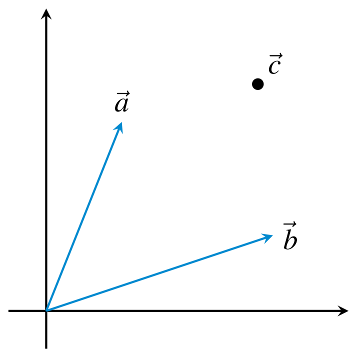
1. Visualizing Span

We are given a point $\vec{c}$ that we want to get to, but we can only move in two directions: $\vec{a}$ and $\vec{b}$. We know that to get to $\vec{c}$, we can travel along $\vec{a}$ for some amount α, then change direction, and travel along $\vec{b}$ for some amount β. We want to find these two scalars α and β, such that we reach point $\vec{c}$. That is, $\alpha\vec{a}+\beta\vec{b}=\vec{c}$.



(a) First, consider the case where $\vec{a}=\begin{bmatrix}1\\0\end{bmatrix}$, $\vec{b}=\begin{bmatrix}1\\1\end{bmatrix}$, and $\vec{c}=\begin{bmatrix}-2\\2\end{bmatrix}$. Find the two scalars α and β, such that we reach point $\vec{c}$. What if $\vec{a}=\begin{bmatrix}1\\1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}2\\1\end{bmatrix}$?

(b) Now formulate the general problem as a system of linear equations and write it in matrix form.

2. Span Proofs

Given some set of vectors $\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}$, show the following:

(a)
$$\text{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}=\text{span}\{\alpha\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\},\ \text{where } \alpha \text{ is a non-zero scalar}$$

In other words, we can scale our spanning vectors and not change their span.

(b)
$$\text{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}=\text{span}\{\vec{v}_2,\vec{v}_1,\ldots,\vec{v}_n\}$$

In other words, we can swap the order of our spanning vectors and not change their span.

(c)
$$\text{span}\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n\}=\text{span}\{\vec{v}_1+\vec{v}_2,\vec{v}_2,\ldots,\vec{v}_n\}$$

In other words, we can add our spanning vectors to one another and not change their span.

3. Four Fundamental Subspaces of a Matrix

We will explore some properties of the four fundamental subspaces of matrices in this problem. Consider the following matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 4 & 0 \\ 3 & 2 & 4 \end{bmatrix}$$

(a) Row reduce the matrix $\mathbf{A}$ above and let this matrix be $\mathbf{A}'$.

(b) Is the column space of $\mathbf{A}$ the same as the column space of $\mathbf{A}'$?

(c) Is the row space of $\mathbf{A}$ the same as the row space of $\mathbf{A}'$?

(d) Do $\mathbf{A}$ and $\mathbf{A}'$ have the same null space?

(e) Which spaces are conserved during row operations?

(f) Which spaces have the same dimensions?

4. Exploring Null Spaces

- The **column space** of a matrix is the **range** or possible outputs of a transformation/function/linear operation. It is also the **span** of the vectors that form the columns of the matrix.
- The **null space** is the set of input vectors that output the zero vector.

For the following five matrices, answer the following questions:

i. What is the column space of $\mathbf{A}$? What is its dimension?

ii. What is the null space of $\mathbf{A}$? What is its dimension?

iii. Are the column spaces of the row reduced matrix $\mathbf{A}$ and the original matrix $\mathbf{A}$ the same?

iv. Do the columns of $\mathbf{A}$ form a basis of $\mathbb{R}^2$ (or $\mathbb{R}^3$ for part (e))? Why or why not?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 4 \\ 3 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & -2 \end{bmatrix}$

5. Row Space

Consider:

$$\mathbf{V} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 0 & 4 \\ 6 & 4 & 10 \\ -2 & 4 & 2 \end{bmatrix}$$

Row reducing this matrix yields:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) Show that the row spaces of **U** and **V** are the same. Argue that in general, Gaussian elimination preserves the row space.

(b) Show that the null spaces of **U** and **V** are the same. Argue that in general, Gaussian elimination preserves the null space.

6. Mechanical Determinants

(a) Compute the determinant of $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

(b) Compute the determinant of $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

(c) Compute the determinant of $\begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 17 & 0 & 0 \\ 0 & 0 & -31 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

7. Row Operations and Determinants

In this question we explore the effect of row operations on the determinant of a matrix. Note that scaling a row by a will increase the determinant by a factor of a, and adding a multiple of one row to another will not change the determinant. Swapping two rows of a matrix and computing the determinant is equivalent to multiplying the determinant of the original matrix by -1. The determinant of an identity matrix is 1. Feel free to prove these properties to convince yourself that they hold for general square matrices.

(a) An upper triangular matrix is a matrix with zeros below its diagonal. For example a 3×3 upper triangular matrix is:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix}$$

By considering row operations and what they do to the determinant, argue that the determinant of a general $n \times n$ upper triangular matrix is the product of its diagonal entries if they are non-zero. For example, the determinant of the 3×3 matrix above is $a_1 \cdot b_2 \cdot c_3$ if $a_1, b_2, c_3 \neq 0$.

(b) If the diagonal of an upper-triangular matrix has a zero entry, argue that its determinant is still the product of its diagonal entries.

8. Inverses

In general, the *inverse* of a matrix "undoes" the operation that the matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$
$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$
$$(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1} \qquad \text{for a nonzero scalar } k$$
$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{assuming } \mathbf{A}, \mathbf{B} \text{ are both invertible}$$

(a) Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$? (Bonus: How does this relate to the invertibility of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$?)
v. Are all square matrices invertible?
vi. How can you find the inverse of a general $n \times n$ matrix?