1. **Coordinate Change Examples**

   (a) **Transformation From Standard Basis To Another Orthonormal Basis in $\mathbb{R}^3$**
   Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

   i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$ where $\vec{x}_u$ contains the coordinates of a vector in a basis of the columns of $\mathbf{U}$ and $\vec{x}_v$ is the coordinates of the same vector in the basis of the columns of $\mathbf{V}$.

   Draw a picture of the two different coordinate frames. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Compute $\vec{x}_v$ and compare the results with your picture. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Are the results intuitive?

   Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. What is $\vec{x}_v$? How would you verify that this is correct?

   (b) **Transformation Between Two Orthonormal Bases in $\mathbb{R}^3$**
   Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix},$$

   i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$. Draw a picture of the two different coordinate frames. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Compute $\vec{x}_v$ and compare the results with your picture. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Are the results intuitive?

   Now let $\vec{x}_u = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. What is $\vec{x}_v$? How would you verify that this is correct?

   (c) What is the coordinate transformation from $\vec{x}_v$ to $\vec{x}_u$, i.e. find $\mathbf{W}$ such $\vec{x}_u = \mathbf{W}\vec{x}_v$?

   (d) **Transformation Between General Bases (Non-Orthogonal) in $\mathbb{R}^2$**
   Calculate the coordinate transformation between the following bases

$$\mathbf{U} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

i.e. find a matrix $\mathbf{T}$, such that $\vec{x}_v = \mathbf{T}\vec{x}_u$. Draw a picture of the two different coordinate frames. Let $\vec{x}_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Compute $\vec{x}_v$ and compare the results with your picture. Repeat this for $\vec{x}_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Are the results intuitive?

Now let $\vec{x}_u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. What is $\vec{x}_v$? How would you verify that this is correct?

2. Module 1 Review

Optional Review Problems:

3. Proofs

(a) Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. Assume that $\mathbf{A}$ is invertible, but $\mathbf{B}$ is not invertible. Show that neither $\mathbf{AB}$ nor $\mathbf{BA}$ is invertible.

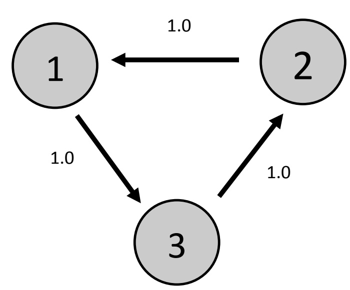
(b) Let $\mathbf{A}$ be an invertible matrix. Show that if λ is an eigenvalue of $\mathbf{A}$, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

4. Justin Beaver (Fall 2015 MT1)

In your homework, there was a question about Justin Bieber's segway — that was about controlling a multi-dimensional system with one control input. In this problem, we will instead think about a curious and superintelligent beaver watching the water level in a pool — this is implicitly about how many sensors are needed to measure the state of a multi-dimensional system.

Three superintelligent rodents live in a network of pools. Justin Beaver lives in pool 1, Selena Gopher lives in pool 2, and Mousey Cyrus lives in pool 3. They are sadly not on talking terms, but Justin really wants to know about the other pools.

Suppose there is a network of pumps connecting the three different pools, given in the figure. $x_1[t], x_2[t]$, and $x_3[t]$ is the water level in each pool at time step t. At each time step, the water from each pool is pumped along the arrows. The water levels are updated according to the matrix:



$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ x_3[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \end{bmatrix}$$

(a) Justin lives in pool 1 so he can watch the water level in this pool. He also knows exactly how the pumps work — i.e. knows the pump matrix $\mathbf{A}$. **Can Justin figure out the initial water levels in all three pools just by watching the water levels in his pool as time goes by? Describe (briefly) in**

words how to do this. How many times does Justin need to observe the water in his own pool to figure this out?

(Hint: No "linear algebra" machinery is needed here. Just think about what Justin observes as time goes by.)

(b) Consider now a general pump matrix $\mathbf{A}$ that is known to Justin, not necessarily the one in the example above. Just for this part, suppose Justin had been told the initial water levels $\vec{x}[0]$ by someone else. Could he figure out $\vec{x}[t]$? **Write an expression for $\vec{x}[t]$ given the initial levels $\vec{x}[0]$ and the pump matrix A.**

(c) Suppose we use $y[t]$ to denote Justin's measurement of the water level in pool 1 at time t. We know that $y[t] = x_1[t]$. **Find a vector $\vec{c}$ such that**

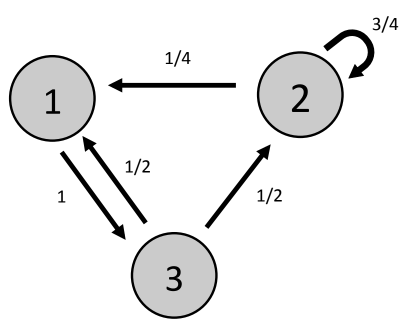
$$y[t] = \vec{c}^T \vec{x}[t]$$

(d) We want to know if tracking the water level in pool 1 is enough to eventually figure out the initial water level in all the pools. First **find a matrix D in terms of $\vec{c}$ and A (and powers of A), such that**

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[T-1] \end{bmatrix} = \begin{bmatrix} & \mathbf{D} & \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \\ x_3[0] \end{bmatrix}$$

(Hint: Think about what the rows of $\mathbf{D}$ *should be. It suffices to give an expression for the jth row* $\mathbf{D}_j$ *of* $\mathbf{D}$*.)*

(e) Now assume we have a specific network of pumps with a different pump matrix.



$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \\ x_3[t+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_1[t] \\ x_2[t] \\ x_3[t] \end{bmatrix}$$

Given this specific $\mathbf{A}$ matrix, **how many time steps T of observations in pool 1 will Justin need in order to recover the initial water levels $\vec{x}[0]$? Argue why this number of observations is enough.**

(f) For the T chosen in the previous part and the pump matrix $\mathbf{A}$ given there, suppose Justin measures

$$y[t] = 1 \qquad \text{for } t = 0, 1, ..., (T-1)$$

What was $\vec{x}[0]$?