Reference Definitions

Inner Product Algebraic definition: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N : \langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^N x_i \cdot y_i$.

Euclidean Norm The *Euclidean Norm* of a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$ is $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$

Vector Scaling Let $c \in \mathbb{R}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$. Recall that $c \cdot \vec{x} = \begin{bmatrix} c \cdot x_1 \\ c \cdot x_2 \\ \vdots \\ c \cdot x_N \end{bmatrix}$.

1. **Lecture Review**

2. **Investigating Inner Products** Now follow your TA as we discover some properties of inner products.

3. **Packings**

   (a) Can three vectors in the $\mathbb{R}^2$ plane have only negative pairwise inner-products? That is, do there exist vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ such that $\langle \vec{u}, \vec{v} \rangle < 0$, $\langle \vec{v}, \vec{w} \rangle < 0$, and $\langle \vec{u}, \vec{w} \rangle < 0$?
   (Hint: Draw a picture!)

   (b) What about four vectors in $\mathbb{R}^2$? That is, do there exist four vectors $\vec{u}, \vec{v}, \vec{w}, \vec{x} \in \mathbb{R}^2$ such that for every pair of vectors $\vec{a}, \vec{b}$: $\left\langle \vec{a}, \vec{b} \right\rangle < 0$?

   (Bonus: What about four vectors in $\mathbb{R}^3$?)

4. **Orthogonal Subspaces**

   Two subspaces $\mathbb{S}_1$ and $\mathbb{S}_2$ of $\mathbb{R}^N$ are said to be orthogonal if all vectors in $\mathbb{S}_1$ are orthogonal to all vectors in $\mathbb{S}_2$. That is,

   $$\langle \vec{v_1}, \vec{v_2} \rangle = 0 \; \forall \vec{v_1} \in \mathbb{S}_1, \vec{v_2} \in \mathbb{S}_2.$$

   (a) Recall that the *row space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the rows of $\mathbf{A}$ and that the *null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\mathbf{A}\vec{v} = \vec{0}$.
   Prove that the row space and null space of any matrix are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}^T) \perp \text{Null}(\mathbf{A}) \; \forall \mathbf{A} \in \mathbb{R}^{M \times N}$.

(b) Recall that the *column space* of an $M \times N$ matrix $\mathbf{A}$ is the subspace spanned by the columns of $\mathbf{A}$ and that the *left null space* of $\mathbf{A}$ is the subspace of all vectors $\vec{v}$ such that $\vec{v}^T\mathbf{A} = \vec{0}^T \iff \mathbf{A}^T\vec{v} = \vec{0}$.
Prove that the column space and left null space of any matrix are orthogonal subspaces. This can be denoted by $\text{Col}(\mathbf{A}) \perp \text{Null}(\mathbf{A}^T)\ \forall \mathbf{A} \in \mathbb{R}^{M\times N}$.

5. Mechanical Projection

In $\mathbb{R}^n$, the projection of vector $\vec{a}$ onto vector $\vec{b}$ is defined as:

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\|\vec{b}\right\|^2}\vec{b} = \frac{\left\langle \vec{a}, \vec{b} \right\rangle}{\left\|\vec{b}\right\|}\hat{b}$$

where $\hat{b}$ is the normalized $\vec{b}$, i.e. a unit vector with the same direction as $\vec{b}$.

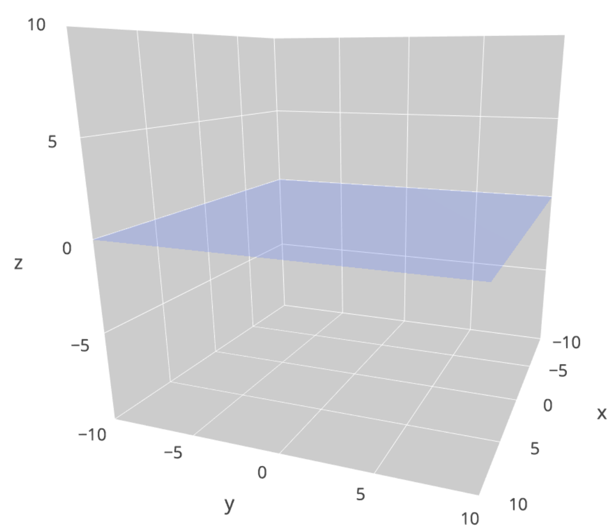
(a) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - that is, onto the x-axis. Graph these two vectors and the projection.

(b) Project $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ onto $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ - that is, onto the y-axis. Graph these two vectors and the projection.

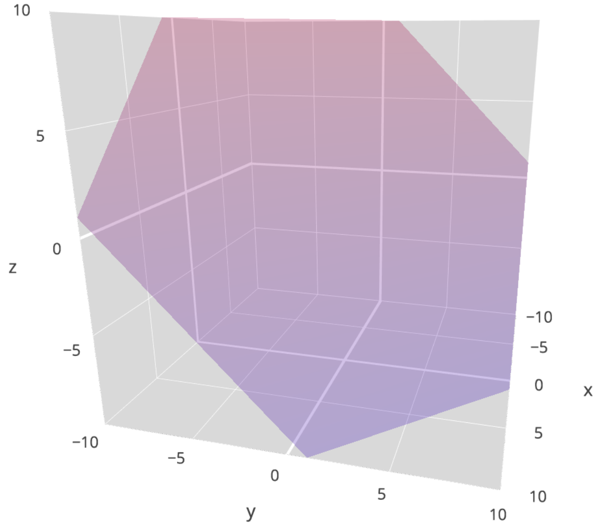
(c) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Graph these two vectors and the projection.

(d) Project $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$ onto $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph these two vectors and the projection.

(e) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the span of the vectors $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ - that is, onto the x-y plane in $\mathbb{R}^3$. Try to visualize how this would appear on the plot below.



(f) Project $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ onto the plane described by $x+y+z=1$. Try to visualize how this would appear on the plot below.



(g) What is the geometric/physical interpretation of projection? Justify using the previous parts.

(h) For the first 4 parts, we looked at two different projections for each vector. For those cases, using only the projected vectors and the vectors we projected onto, do we have enough information to reconstruct the original vector?

(i) Given information about n projections of a vector in $\mathbb{R}^n$, when do we have enough information to reconstruct the original vector? Always? Never?