EECS 16A Designing Information Devices and Systems I
Summer 2017 D. Aranki, F. Maksimovic, V. Swamy Discussion 7D

1. Orthogonal Matching Pursuit Lecture

Orthogonal Matching Pursuit (OMP) algorithm:

Inputs:
- An $n \times d$ dimension signature matrix $\mathbf{S}$ with columns $\vec{S}_i$
- An n dimension measurement vector $\vec{y}$
- The sparsity level m of the signal

Outputs:
- An estimate $\hat{\vec{x}}$ in $\mathbb{R}^d$ of the ideal message
- A set Λ_m containing m elements from $\{1, 2, \dots, d\}$
- An n-dimensional approximation $\hat{\vec{y}}$ of the measurement vector $\vec{y}$
- An n-dimensional residual $\vec{r} = \vec{y} - \hat{\vec{y}}$

procedure OMP($\mathbf{S}$, $\vec{y}$, m)
  $\vec{r} \leftarrow \vec{y}$ (portion of measurement to be explained)
  $\mathbf{A}_0 \leftarrow [\,]$ (signatures found in the measurement)
  $j \leftarrow 1$ (iteration variable)
  $\Lambda_0 \leftarrow [\,]$ (indices of vectors found)
  while $\vec{r} \neq \vec{0}$ **do**
    procedure FIND SIGNATURE($\vec{r}$, $\mathbf{S}$)
      $k \leftarrow \text{argmax}_i \langle \vec{S}_i, \vec{r} \rangle$
      $\mathbf{A}_j \leftarrow \begin{bmatrix} \mathbf{A}_{j-1} & \vec{S}_k \end{bmatrix}$
      $\Lambda_j \leftarrow \Lambda_{j-1} \cup k$
    end procedure
    procedure FIND PROJECTION($\mathbf{A}_j$, $\vec{y}$)
      $\hat{\vec{x}}_j = (\mathbf{A}_j^T \mathbf{A}_j)^{-1} \mathbf{A}_j^T \vec{y}$
      $\hat{\vec{y}}_j = \mathbf{A}_j \hat{\vec{x}}_j$
      $\vec{r} \leftarrow \vec{y} - \hat{\vec{y}}$
    end procedure
    $j \leftarrow j + 1$
  end while
  The estimate for the ideal signal has non-zero indices at the components listed in Λ_m. The value of $\vec{x}$ in component λ_j equals the jth component of $\hat{\vec{x}}$.
end procedure

2. One Magical Procedure (Fall 2015 Final)

Suppose that we have a vector $\vec{x} \in \mathbb{R}^5$ and an $N \times 5$ measurement matrix $\mathbf{M}$ defined by column vectors

$\vec{c}_1, \dots, \vec{c}_5$, such that:

$$\mathbf{M}\vec{x} = \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_5 \\ | & & | \end{bmatrix} \vec{x} \approx \vec{b}$$

We can treat the vector $\vec{b} \in \mathbb{R}^N$ as a noisy measurement of the vector $\vec{x}$, with measurement matrix $\mathbf{M}$ and some additional noise in it as well.

You also know that the true $\vec{x}$ is sparse – it only has two non-zero entries and all the rest of the entries are zero in reality. Our goal is to recover this original $\vec{x}$ as best we can.

However, your intern has managed to lose not only the measurements $\vec{b}$ but the entire measurement matrix $\mathbf{M}$ as well!

Fortunately, you have found a backup in which you have all the pairwise inner products $\langle \vec{c}_i, \vec{c}_j \rangle$ between the columns of $\mathbf{M}$ and each other as well as all the inner products $\left\langle \vec{c}_i, \vec{b} \right\rangle$ between the columns of $\mathbf{M}$ and the vector $\vec{b}$. Finally, you also know the inner product $\left\langle \vec{b}, \vec{b} \right\rangle$ of $\vec{b}$ with itself.

All the information you have is captured in the following table of inner products. (These are not the vectors themselves.)

$\langle \cdot, \cdot \rangle$	$\vec{c}_1$	$\vec{c}_2$	$\vec{c}_3$	$\vec{c}_4$	$\vec{c}_5$	$\vec{b}$
$\vec{c}_1$	2	0	1	-1	1	1
$\vec{c}_2$		2	1	-1	-1	-5
$\vec{c}_3$			2	0	-1	2
$\vec{c}_4$				2	-1	6
$\vec{c}_5$					2	-1
$\vec{b}$						29

(So, for example, if you read this table, you will see that the inner product $\langle \vec{c}_2, \vec{c}_3 \rangle = 1$, that the inner product $\left\langle \vec{c}_3, \vec{b} \right\rangle = 2$, and that the inner product $\left\langle \vec{b}, \vec{b} \right\rangle = 29$. By symmetry of the real inner product, $\langle \vec{c}_3, \vec{c}_2 \rangle = 1$ as well.)

Your goal is to find which entries of $\vec{x}$ are non-zero and what their values are.

(a) Use the information in the table above to answer which of the $\vec{c}_1, \dots, \vec{c}_5$ has the largest magnitude inner product with $\vec{b}$?

(b) Let the vector with the largest magnitude inner product with $\vec{b}$ be $\vec{c}_a$. Let $\vec{b}_p$ be the projection of $\vec{b}$ onto $\vec{c}_a$. Write $\vec{b}_p$ symbolically as an expression only involving $\vec{c}_a$, $\vec{b}$, and their inner products with themselves and each other.

(c) Use the information in the table above to find which of the column vectors $\vec{c}_1, \dots, \vec{c}_5$ has the largest magnitude inner product with the residue $\vec{b} - \vec{b}_p$.

Hint: The linearity of inner products might prove useful.

(d) Suppose that the vectors we found in parts (a) and (c) are $\vec{c}_a$ and $\vec{c}_c$. These correspond to the components of $\vec{x}$ that are non-zero, that is, $\vec{b} \approx x_a \vec{c}_a + x_c \vec{c}_c$. However, there might be noise in the measurements $\vec{b}$, so we want to find the linear least squares estimates $\widehat{x}_a$ and $\widehat{x}_c$. Write a matrix expression for $\begin{bmatrix} \widehat{x}_a \\ \widehat{x}_c \end{bmatrix}$ in terms of appropriate matrices filled with the inner products of $\vec{c}_a$, $\vec{c}_c$, $\vec{b}$.

(e) Compute the numerical values of $\widehat{x}_a$ and $\widehat{x}_c$ using the information in the table.