1. Mechanical Gram-Schmidt (Fall 2016 Final)

(a) Use Gram-Schmidt to find an orthonormal basis for the following three vectors.

$$\vec{v}_1 = \begin{bmatrix} \sqrt{2} \\ -\sqrt{2} \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \sqrt{2} \\ 0 \\ -\sqrt{2} \end{bmatrix}$$

(b) Express $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ as vectors in the basis you found in part **??**.

2. Gram-Schmidt Properties

(a) If we are performing the Gram-Schmidt method on a set of vectors, does the order in which we take the vectors matter? Consider the following set of vectors.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Perform Gram-Schmidt on these vectors first in the order $\vec{v}_1,\ \vec{v}_2,\ \vec{v}_3$ and then in the order $\vec{v}_3,\ \vec{v}_2,\ \vec{v}_1$. Do you get the same answer?

(b) What happens when we perform Gram-Schmidt on a set of n vectors $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$, where only $n-1$ of them are linearly independent?

3. Orthogonal Polynomials

So far we've applied most of the linear algebra tools learned to vector spaces over real numbers, that is $\mathbb{R}^n$. However, the Gram-Schmidt process and least squares can be applied to other vector spaces as well. Here, we consider the polynomial vector space $\mathbb{P}^n$.

Suppose we are operating in $\mathbb{P}^3$, that is polynomials of degree 3 or less.

(a) Represent the polynomial $x^3 + 5x^2 + 3x + 2$ as a linear combination of the standard basis vectors.

(b) There are other basis for the $\mathbb{P}^3$. Represent $x^3 + 5x^2 + 3x + 2$ as a linear combination of the following basis: $\{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1, 1\}$.

(c) We cannot apply the same inner product we use over $\mathbb{R}^n$ over $\mathbb{P}^n$. Instead, we have to create a new inner product. One inner product is: $\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)dx$. Show that this inner product satisfies the properties below:

$$\langle f(x), g(x) \rangle = \langle g(x), f(x) \rangle$$
$$\langle cf(x), g(x) \rangle = c\langle f(x), g(x) \rangle$$
$$\langle f(x) + g(x), h(x) \rangle = \langle f(x), h(x) \rangle + \langle g(x), h(x) \rangle$$

(d) With the definition of inner product shown above, we can find an orthonormal basis for polynomials using the Gram-Schmidt process. Find an orthonormal basis for $\mathbb{P}^3$.

(e) We can also use this definition to find a least squares approximation. Suppose we want to find a polynomial approximation for the function $f(x) = e^x$. Find the best approximation for e^x in $\mathbb{P}^3$. You may find the following integrals useful:

$$\int xe^x dx = (x-1)e^x$$
$$\int x^2 e^x dx = (x^2 - 2x + 2)e^x$$
$$\int x^3 e^x dx = (x^3 - 3x^2 + 6x - 6)e^x$$