EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 0C

1. Linear or Nonlinear

Determine whether the following functions (f: $\mathbb{R}^2 \to \mathbb{R}$) are linear or nonlinear.

(a)

$$f(x_1,x_2) = 3x_1 + 4x_2$$

Answer: To check for linearity, check for superposition (additivity) and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1,x_2) + \beta f(y_1,y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Linear

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= 3(\alpha x_1 + \beta y_1) + 4(\alpha x_2 + \beta y_2) \\ &= \alpha(3x_1 + 4x_2) + \beta(3y_1 + 4y_2) \\ &= \alpha f(x_1,x_2) + \beta f(y_1,y_2) \end{aligned}$$

Alternatively you can state that this function is linear because it is of the form:

$$f(x_1,x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(b)

$$f(x_1,x_2) = e^{x_2} + x_1^2$$

Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling).
In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1,x_2) + \beta f(y_1,y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= e^{\alpha x_2 + \beta y_2} + (\alpha x_1 + \beta y_1)^2 \\ &\neq e^{\alpha x_2} + (\alpha x_1)^2 + e^{\beta y_2} + (\beta y_1)^2 \\ &= \alpha f(x_1,x_2) + \beta f(y_1,y_2) \end{aligned}$$

Alternatively you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1,x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

(c)

$$f(x_1,x_2) = x_2 - x_1 + 3$$

Answer: To check for linearity, check for additivity and homogeneity (multiplicative scaling). In other words we must check that:

$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) = \alpha f(x_1,x_2) + \beta f(y_1,y_2) \quad \forall \alpha, \beta, x_1, x_2, y_1, y_2 \in \mathbb{R}$$

Nonlinear, in fact this function is affine (see notes for more details).
You may simply state that this function doesn't satisfy homogeneity when scaled by 0.
Alternatively you can show:

$$\begin{aligned} f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) &= (\alpha x_2 + \beta y_2) - (\alpha x_1 + \beta y_1) + 3 \\ &\neq \alpha(x_2 - x_1 + 3) + \beta(y_2 - y_1 + 3) \\ &= \alpha f(x_1,x_2) + \beta f(y_1,y_2) \end{aligned}$$

Alternatively you can state that this function is nonlinear because it is NOT of the form:

$$f(x_1,x_2) = a_1 x_1 + a_2 x_2$$

where a_1 and a_2 are constants.

2. Gaussian Elimination

Use Gaussian elimination to solve the following systems. Does a solution exist? Is it unique?

(a)

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

Answer:

The solution is not unique. One solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 4 & 2 \\ 1 & 2 & 8 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

Answer:

No solution. When you do Gaussian elimination, you will get a row that looks like $\left[\begin{array}{ccc|c} 0 & 0 & 0 & a \end{array}\right]$, where $a \neq 0$ in the augmented matrix.

(d)

$$\begin{bmatrix} 2 & 2 & 3 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Answer:

There are many solutions. One solutions is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

Answer:
No solution.

3. Finding The Bright Cave

Nara the one-handed druid and Kody the one-handed ranger find themselves in dire straits. Before them is a cliff with four cave entrances arranged in a square: two upper caves and two lower caves. Each entrance emits a certain amount of light, and the two wish to find exactly the amount of light coming from each cave. Here's the catch: after contracting a particularly potent strain of ghoul fever, our intrepid heroes are only able to see the total intensity of light before them (so their eyes operate like a single-pixel camera). Kody and Nara are capable adventurers, but they don't know any linear algebra – and they need your help.

Kody proposes an imaging strategy where he uses his hand to completely block the light from two caves at a time. He is able to take measurements using the following four masks (black means the light is blocked from that cave):

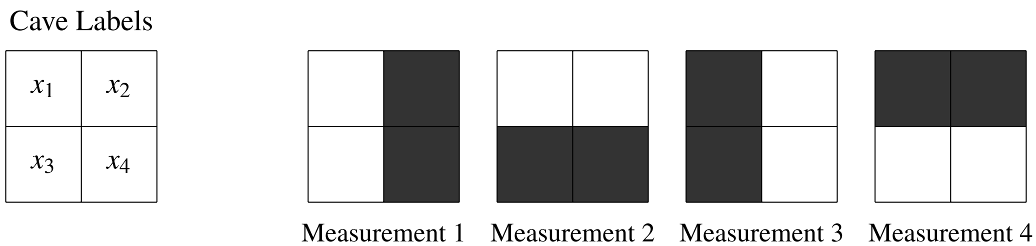

Figure 1: Four image masks.

(a) Let x_1, x_2, x_3, and x_4 represent the magnitude of light emanating from the four cave entrances shown in the image above. Write an equation for each masking process in Figure 1 which results in the four measurements of total light: m_1, m_2, m_3, and m_4.

Answer:

$$m_1 = x_1 + x_3$$
$$m_2 = x_1 + x_2$$
$$m_3 = x_2 + x_4$$
$$m_4 = x_3 + x_4$$

(b) Does Kody's set of masks give us a unique solution for all four caves' light intensities? Why or why not?

Answer:

Notice the equations. If we find that we could get one equation from the other equations, then we know that the solution is not unique. Notice that the sum of the first and the third row is the same is the sum of the second and fourth row.

$$m_1 + m_3 = m_2 + m_4$$
$$m_4 = m_1 + m_3 - m_2$$
$$(x_3 + x_4) = (x_1 + x_3) + (x_2 + x_4) - (x_1 + x_2)$$
$$x_3 + x_4 = x_3 + x_4$$

(c) Nara, in her infinite wisdom, places her one hand diagonally across the entrances, covering two of the cave entrances. However, her hand is not wide enough, letting in 50% of the light from the caves covered and 100% of the light from the caves not covered. The following diagram shows the percentage of light let through from each cave:

50%	100%
100%	50%

Does this additional measurement give them enough information to solve the problem? Why or why not?

Answer:

The answer is yes; the additional measurement does give them enough information to solve the problem. Since Nara's measurement provides additional information, we are now able to solve for all four light intensities uniquely.

This can be shown using algebra with the addition of the following measurement:

$$m_5 = \frac{1}{2}x_1 + x_2 + x_3 + \frac{1}{2}x_4$$

Note that we can isolate x_3 by combining measurements 2, 3, and 5:

$$x_3 = m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3$$

We can use further substitution to determine x_1, x_2, and x_4:

$$x_1 = m_1 - x_3 = m_1 - (m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3) = m_1 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3$$
$$x_2 = m_2 - x_1 = m_2 - (m_1 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3) = -m_1 + m_5 + \frac{1}{2}m_2 - \frac{1}{2}m_3$$
$$x_4 = m_4 - x_3 = m_4 - (m_5 - \frac{1}{2}m_2 - \frac{1}{2}m_3) = m_4 - m_5 + \frac{1}{2}m_2 + \frac{1}{2}m_3$$