EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 0D

1. Solving Systems of Equations

(a) Systems of linear equations can either have one solution, an infinite number of solutions, or no solution at all. For the following system of equations, state whether or not a solution exists. If a solution exists, list all of them.

i. $$\begin{cases} 49x + 7y = 49 \\ 42x + 6y = 42 \end{cases}$$

Answer:

$$\begin{cases} 49x + 7y = 49 \\ 42x + 6y = 42 \end{cases} \rightarrow \begin{cases} 0 = 0 \\ 42x + 6y = 42 \end{cases} \quad \text{using } R_1 \leftarrow -\tfrac{7}{6}R_2 + R_1$$

$$\rightarrow \begin{cases} 0 = 0 \\ 7x + y = 7 \end{cases} \quad \text{using } R_2 \leftarrow \tfrac{1}{6}R_2$$

Let x be a free variable, such as a. Set x = a, solve for y in terms of a.

$$x = a$$
$$7a + y = 7$$
$$y = 7 - 7a$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 7-7a \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} + \begin{bmatrix} a \\ -7a \end{bmatrix}, \forall a \in \mathbb{R}$$

ii. $$\begin{cases} 5x + 3y = -21 \\ 2x + y = -9 \end{cases}$$

Answer:

$$\begin{cases} 5x + 3y = -21 \\ 2x + y = -9 \end{cases} \rightarrow \begin{cases} -x = 6 \\ 2x + y = -9 \end{cases} \quad \text{using } R_1 \leftarrow -3R_2 + R_1$$

$$\rightarrow \begin{cases} -x = 6 \\ y = 3 \end{cases} \quad \text{using } R_2 \leftarrow 2R_1 + R_2$$

$$\rightarrow \begin{cases} x = -6 \\ y = 3 \end{cases} \quad \text{using } R_1 \leftarrow -R_1$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$$

iii. $$\begin{cases} 49x + 7y = 60 \\ 42x + 6y = 30 \end{cases}$$

Answer:

$$\begin{cases} 49x + 7y = 60 \\ 42x + 6y = 30 \end{cases} \to \begin{cases} 0 = 25 \\ 42x + 6y = 30 \end{cases} \qquad \text{using } R_1 \leftarrow -\tfrac{7}{6}R_2 + R_1$$

No solution

iv. $$\begin{cases} 2x + 2y + 4z = -1 \\ y + z = -2 \\ x + 2y + 3z = 2 \end{cases}$$

Answer:

$$\begin{cases} 2x + 2y + 4z = -1 \\ y + z = -2 \\ x + 2y + 3z = 2 \end{cases} \to \begin{cases} -2y - 2z = -5 \\ y + z = -2 \\ x + 2y + 3z = 2 \end{cases} \qquad \text{using } R_1 \leftarrow -2R_3 + R_1$$

$$\to \begin{cases} 0 = -1 \\ y + z = -2 \\ x + 2y + 3z = 2 \end{cases} \qquad \text{using } R_1 \leftarrow 2R_2 + R_1$$

No solution

v. $$\begin{cases} 2x + 2y + 4z = 6 \\ y + z = 1 \\ x + 2y + 3z = 4 \end{cases}$$

Answer:

$$\begin{cases} 2x + 2y + 4z = 6 \\ y + z = 1 \\ x + 2y + 3z = 4 \end{cases} \to \begin{cases} -2y - 2z = -2 \\ y + z = 1 \\ x + 2y + 3z = 4 \end{cases} \qquad \text{using } R_1 \leftarrow -2R_3 + R_1$$

$$\to \begin{cases} 0 = 0 \\ y + z = 1 \\ x + 2y + 3z = 4 \end{cases} \qquad \text{using } R_1 \leftarrow -2R_2 + R_1$$

$$\to \begin{cases} 0 = 0 \\ y + z = 1 \\ x + z = 2 \end{cases} \qquad \text{using } R_3 \leftarrow -2R_2 + R_3$$

Let z be a free variable, such as -a. Set z = -a, solve for x and y in terms of -a.

$$z = -a$$
$$x = 2 + a$$
$$y = 1 + a$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2+a \\ 1+a \\ -a \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ a \\ -a \end{bmatrix}, \forall a \in \mathbb{R}$$

vi. $$\begin{cases} x + y + z = 4 \\ 3z = 6 \\ y + z = 3 \end{cases}$$

Answer:

$$\begin{cases} x + y + z = 4 \\ 3z = 6 \\ y + z = 3 \end{cases} \rightarrow \begin{cases} x + y + z = 4 \\ z = 2 \\ y + z = 3 \end{cases} \quad \text{using } R_2 \leftarrow \tfrac{1}{3}R_2$$

$$\rightarrow \begin{cases} x + y + z = 4 \\ z = 2 \\ y = 1 \end{cases} \quad \text{using } R_3 \leftarrow R_3 - R_2$$

$$\rightarrow \begin{cases} x = 1 \\ z = 2 \\ y = 1 \end{cases} \quad \text{using } R_1 \leftarrow R_1 - R_2 - R_3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(b) Systems of equations can also be interpreted graphically. We will try to build a graphical intuition for the results you found in the previous part. Follow along as your TA walks through `dis0D.ipynb`.

Answer:

i. The lines lie on top of one another (i.e. they are the same line), so there are an infinite number of solutions. This system is referred to as underdetermined, which means that there are more unknowns than equations. Though it appears we have two equations and two unknowns, dividing the top equation by 7 and the bottom one by 6 quickly reveals that they are both the same equation.

ii. The lines intersect at one point, so the solution is unique.

iii. The lines do not intersect (a little algebraic manipulation on the equations would reveal that the two lines are parallel). There is no solution.

iv. The intersection of the planes is null. There is no solution.

v. The intersection of the planes is a line, so there is an infinite number of solutions. This system is also underdetermined. Subtracting the second equation solution from the third and multiplying the result by two yields the first equation. In other words including the first equation is redundant because any point that satisfies the second and third equation will certainly satisfy the first. In effect, we have two equations and one unknown.

vi. The intersection of the planes is a single point, so there is a unique solution.

2. Vectors Introduction to vectors and vector addition.

Definitions:
Vector: An ordered list of elements - for example:

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$$

$\mathbb{R}$ or $\mathbb{R}^1$: The set of all real numbers (i.e. the real line)
$\mathbb{R}^2$: The set of all two-element vectors with real numbered entries (i.e. plane of 2×1 vectors) - for example:

$$\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \in \mathbb{R}^2$$

$\mathbb{R}^3$: The set of all three-element vectors with real numbered entries (i.e. 3-space of 3×1 vectors) - for example:

$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3$$

$\mathbb{R}^n$: The set of all n-element vectors with real numbered entries (i.e. n-space of $n \times 1$ vectors)

(a) Are the following vectors in $\mathbb{R}^2$?

i. $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

ii. $\begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}$

Answer:

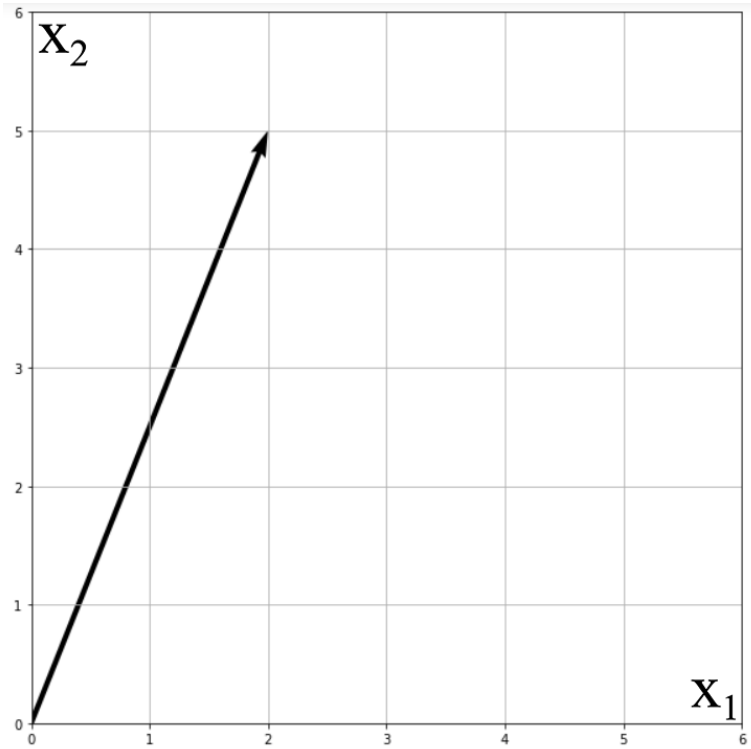
i. Yes, it is a two element vector of real numbered entries.
ii. No, it is a three element vector of real numbered entries.

(b) Graphically show the vectors:

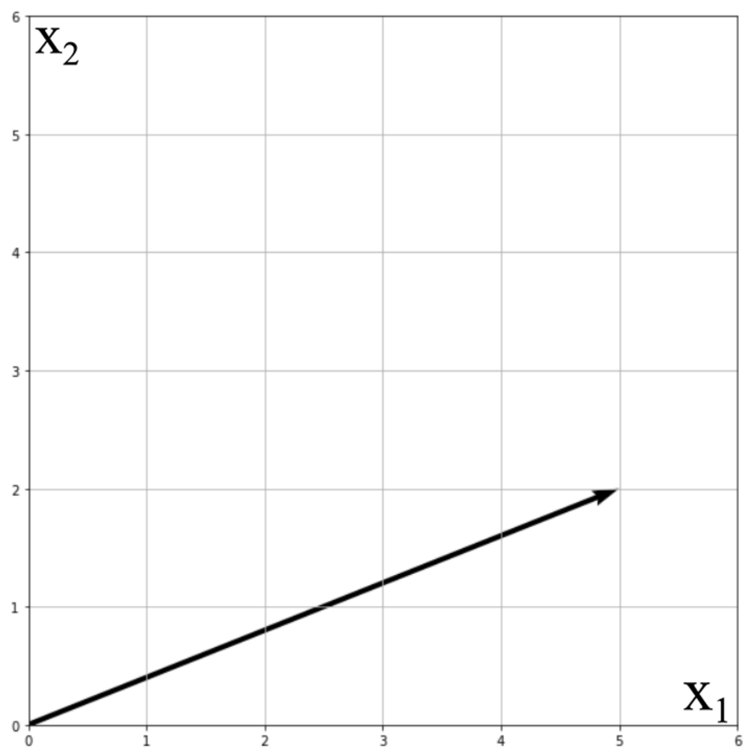
i. $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

ii. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Answer:



i.

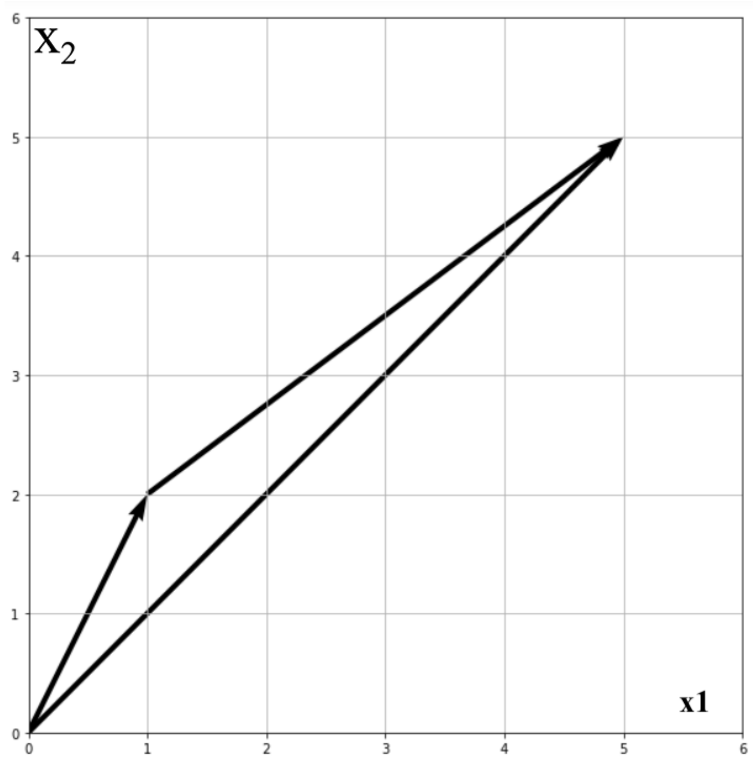


ii.

(c) Graphically show the vector sum and check your answer algebraically:

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}$$

Answer:



$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}5\\5\end{bmatrix}$$

3. Matrix Multiplication

Consider the following matrices:

$$\mathbf{A_1}=\begin{bmatrix}1 & 4\end{bmatrix}\quad \mathbf{B_1}=\begin{bmatrix}3\\2\end{bmatrix}\quad \mathbf{A}=\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix}\quad \mathbf{B}=\begin{bmatrix}3 & 2\\2 & 1\end{bmatrix}\quad \mathbf{C}=\begin{bmatrix}1 & 9 & 5 & 7\\4 & 3 & 2 & 2\end{bmatrix}\quad \mathbf{D}=\begin{bmatrix}5 & 5 & 8\\6 & 1 & 2\\4 & 1 & 7\\3 & 2 & 2\end{bmatrix}\quad \mathbf{E}=\begin{bmatrix}8 & 1 & 6\\3 & 5 & 7\\4 & 9 & 2\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}5 & 3 & 4\\1 & 8 & 2\\2 & 3 & 5\end{bmatrix}$$

For each matrix multiplication problem, if the product exists, find the product by hand. Otherwise, explain why the product does not exist.

(a) $\mathbf{A_1B_1}$

Answer: A_1 is a 1×2 vector and B_1 is a 2×1 vector, so the product exists. $\mathbf{A_1B_1}=11$.

(b) $\mathbf{AB}$

Answer: Since both $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, the product exists and is a 2×2 matrix.

$\mathbf{AB}=\begin{bmatrix}11 & 6\\12 & 7\end{bmatrix}$.

(c) $\mathbf{BA}$

Answer: Since both $\mathbf{A}$ and $\mathbf{B}$ are 2×2 matrices, the product exists and is a 2×2 matrix.

$\mathbf{BA}=\begin{bmatrix}7 & 18\\4 & 11\end{bmatrix}$.

(d) **AC**

Answer: Since **A** is a 2×2 matrix and **C** is a 2×4 matrix, the product exists and is a 2×4 matrix.

$$\mathbf{AC} = \begin{bmatrix} 17 & 21 & 13 & 15 \\ 14 & 27 & 16 & 20 \end{bmatrix}.$$

(e) **DC**

Answer: Since **C** is a 2×4 matrix and **D** is a 4×3 matrix, the product does not exist. This is because the number of columns in the first matrix (**D**) should match the number of rows in the second matrix (**C**) for this product to be defined.

(f) **CD** (Write down the dimensions of the product if it exists. For practice, you can compute the product on your own)

Answer: Since **C** is a 2×4 matrix and **D** is a 4×3 matrix, the product exists and is a 2×3 matrix.

$$\mathbf{CD} = \begin{bmatrix} 100 & 33 & 75 \\ 52 & 29 & 56 \end{bmatrix}.$$

(g) **EF** (Practice on your own)

Answer: Since **E** and **F** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{EF} = \begin{bmatrix} 53 & 50 & 64 \\ 34 & 70 & 57 \\ 33 & 90 & 44 \end{bmatrix}.$$

(h) **FE** (Practice on your own)

Answer: Since **E** and **F** are both 3×3 matrices, the product exists and is another 3×3 matrix.

$$\mathbf{FE} = \begin{bmatrix} 65 & 56 & 59 \\ 40 & 59 & 66 \\ 45 & 62 & 43 \end{bmatrix}.$$