EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 1D

1. Inverses

In general, the *inverse* of a matrix "undoes" the operation that the matrix performs. Mathematically, we write this as

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I},$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Intuitively, this means that applying a matrix to a vector and then subsequently applying its inverse is the same as leaving the vector untouched.

Properties of Inverses

For a matrix $\mathbf{A}$, if its inverse exists, then:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$
$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$
$$(k\mathbf{A})^{-1} = k^{-1}\mathbf{A}^{-1} \qquad \text{for a nonzero scalar } k$$
$$\left(\mathbf{A}^{T}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{T}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad \text{assuming } \mathbf{A},\mathbf{B} \text{ are both invertible}$$

(a) Prove that $(\mathbf{ABC})^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}$.

Answer:

$$\begin{aligned}\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{ABC} &= \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{IBC} \\ &= \mathbf{C}^{-1}\mathbf{IC} \\ &= \mathbf{I}\end{aligned}$$

(b) Now consider the following four matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

i. What do each of these matrices do when you multiply them by a vector $\vec{x}$? Draw a diagram.
ii. Intuitively, can these operations be undone? Why or why not? Make an intuitive argument.
iii. Are the matrices $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$ invertible?
iv. Can you find anything in common about the rows (and columns) of $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$? (*Bonus:* How does this relate to the invertibility of $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}$?)
v. Are all square matrices invertible?
vi. How can you find the inverse of a general $n \times n$ matrix?

Answer:

i. • $\mathbf{A}$: Keeps the x component and throws away the y component.

- **B**: Keeps the y component and throws away the x component.
- **C**: Replaces the x and y components with the average of the x and y components.
- **D**: Finds a weighted sum of the x and y components. Places the sum in x and twice the sum in y.

ii. Intuitively, none of these operations can be undone because we lost some information. In the first two, we lost one component of the original. In the third case, we replaced both x and y with the average of the two. Thus, different inputs could lead to the same average and we wouldn't be to tell them apart. In the fourth case, we took a weighted sum of the x and y components. There are different values for x and y that could lead to the same sum. However, we cannot recover the original x and y because we didn't compute two unique weighted sums. Instead, we just multiplied the sum by two for the y component of the output.

iii. Since the operations are not one-to-one reversible, $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are not invertible.

iv. The rows of $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are all linearly dependent. The same is true for the columns. The generalization is that if a matrix is not invertible, then its rows and columns will be linearly dependent.

v. No. We have seen in the above parts that there are square matrices that are not invertible.

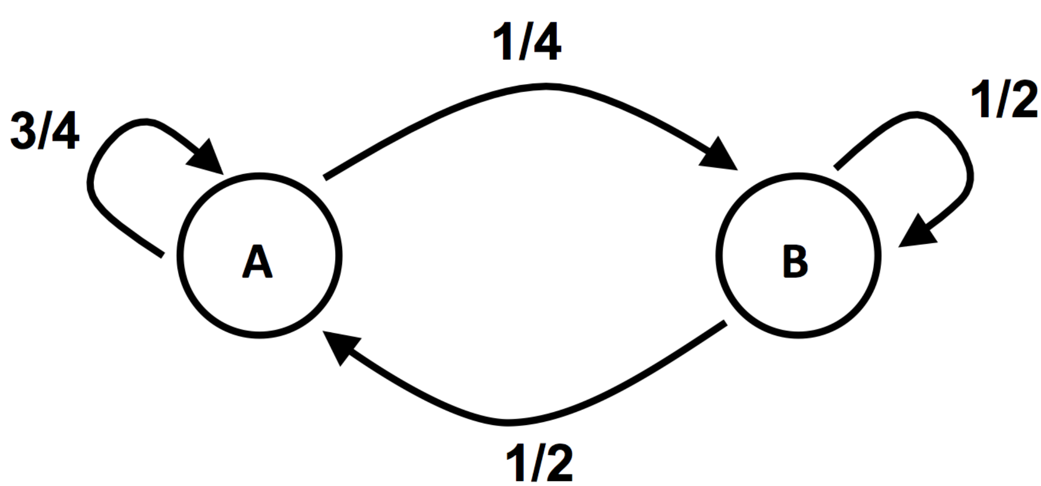
vi. We know that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. If we treat this as our now familiar $\mathbf{A}\vec{x} = \vec{b}$, we can use Gaussian elimination:

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{I} \end{array}\right] \implies \left[\begin{array}{c|c} \mathbf{I} & \mathbf{A}^{-1} \end{array}\right]$$

2. Transition Matrix

(a) Suppose there exists some network of pumps as shown in the diagram below. Let $\vec{x}(n) = \begin{bmatrix} x_A(n) \\ x_B(n) \end{bmatrix}$ where $x_A(n)$ and $x_B(n)$ are the states at timestep n.

Find the state transition matrix S, such that $\vec{x}(n+1) = S\vec{x}(n)$.



Answer:

We can write the following equations by examining the state transition diagram:

$x_A(n+1) = (3/4)x_A(n) + (1/2)x_B(n)$

$x_B(n+1) = (1/4)x_A(n) + (1/2)x_B(n)$.

From here, we can directly write down the state transition matrix as $S = \begin{bmatrix} 3/4 & 1/2 \\ 1/4 & 1/2 \end{bmatrix}$.

(b) Let us now find the matrix S^{-1} such that we can recover $\vec{x}(n-1)$ from $\vec{x}(n)$. Specifically, solve for S^{-1} such that $\vec{x}(n-1) = S^{-1}\vec{x}(n)$.

Answer:

We can use Gaussian elimination to solve for the matrix S^{-1}, i.e. inverse of the matrix S that we just found:

$$\left[\begin{array}{cc|cc} 3/4 & 1/2 & 1 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow \frac{4}{3}R_1} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 1/4 & 1/2 & 0 & 1 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow -4R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ -1 & -2 & 0 & -4 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -4/3 & 4/3 & -4 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 2/3 & 4/3 & 0 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \overbrace{\Rightarrow}^{R_1 \leftarrow R_1 + R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & -2/3 & 2/3 & -2 \end{array}\right] \overbrace{\Rightarrow}^{R_2 \leftarrow \frac{-3}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & 3 \end{array}\right].$$

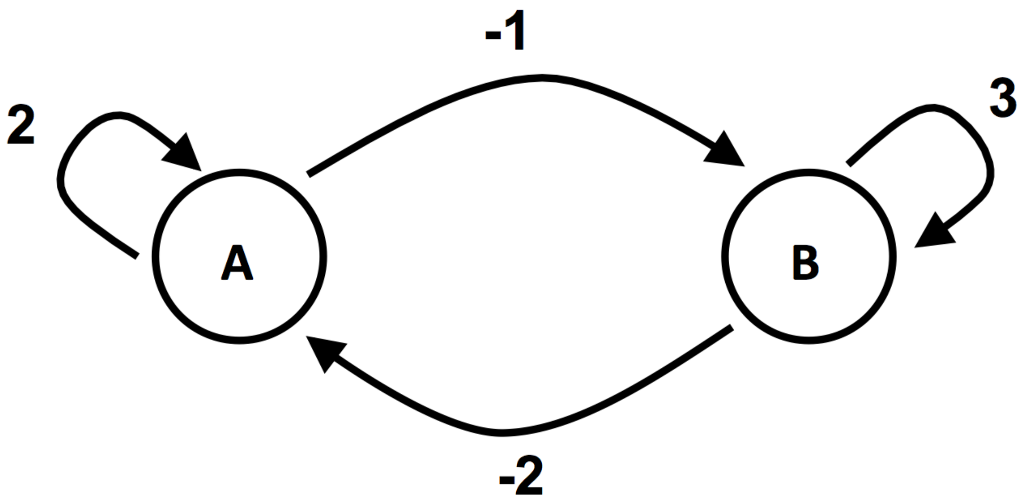
Therefore:

$S^{-1} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$.

Note that the columns of S^{-1} still sum to 1.

(c) Now draw the state transition diagram that corresponds to the S^{-1} that you just found.

Answer:

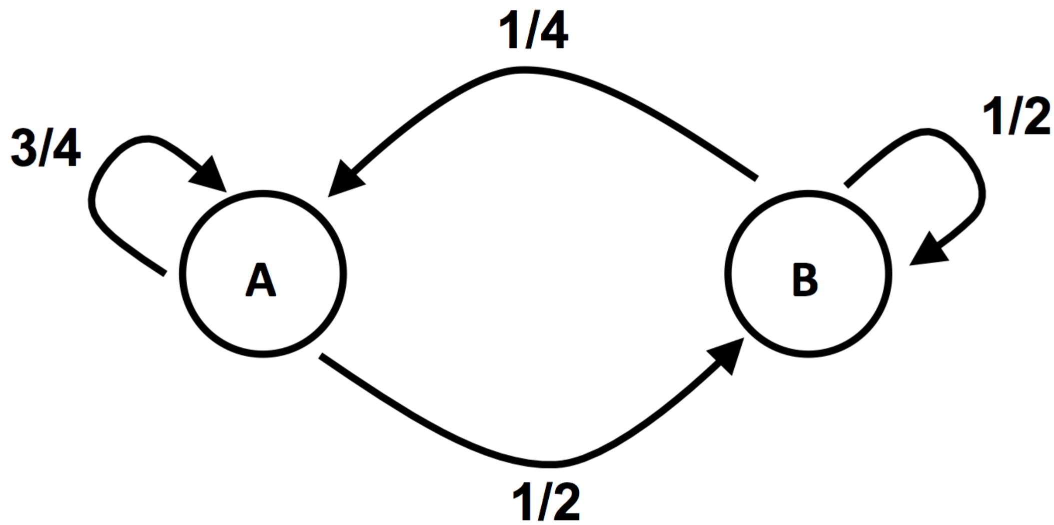


We can write the following equations by examining the state transition diagram:
$x_A(n-1) = 2x_A(n) - 2x_B(n)$

$x_B(n-1) = -x_A(n) + 3x_B(n)$.

Because the matrix S^{-1} is an inverse matrix, it can be thought of as the matrix that *turns back time* for the pump system. Although it is non-physical, the weights that have an absolute value greater than 1 can be thought of as "generating" water, and the weights that have negative weight can be thought of as "destroying" water. However, note that the outflow weights of each node still sum to 1 (i.e. the columns of S^{-1} still sum to 1). This means that in total all of the water is being conserved during the transition between time steps, even when time is reversed.

(d) Redraw the diagram from the first part of the problem, but now with the directions of the arrows reversed. Let us call the state transmission matrix of this "reversed" state transition diagram T. Does $T = S^{-1}$?



Answer:

After drawing the "reversed" state transition diagram, we can write the following equations:
$x_A(n+1) = (3/4)x_A(n) + (1/4)x_B(n)$

$x_B(n+1) = (1/2)x_A(n) + (1/2)x_B(n)$.

From here, we can directly write down the state transition matrix as: $T = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$.

Note that $T \neq S^{-1}$. What we have actually found is that T is equal to the *transpose* of S, denoted by S^T (the superscript T denotes the transpose of a matrix). The transpose of a matrix is when its rows become its columns. In general, a matrix's inverse and its transpose are not equal to each other.

3. Identifying a Basis

Does each of these sets of vectors describe a basis for $\mathbb{R}^3$? If the vectors do not form a basis for $\mathbb{R}^3$, can they be thought of as a basis for some other vector space? If so, write an expression describing this vector space.

$$V_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad V_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad V_3 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Answer:

- V_1: The vectors are linearly independent, but they are not a basis for $\mathbb{R}^3$, because you cannot construct all vectors in $\mathbb{R}^3$ using these vectors. Instead, they are a basis for some 2-dimensional subspace of $\mathbb{R}^3$. This subspace can be described by span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.
- V_2: Yes, the vectors are linearly independent and will form a basis for $\mathbb{R}^3$. To check that the vectors are linearly independent, you should do Gaussian Elimination of the matrix of the columns: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$. Check that you can get all the way to identity, i.e. the system has a unique solution.

- V_3: No, $\vec{v}_2 + \vec{v}_3 = \vec{v}_1$, so the vectors are linearly dependent. Hence, they cannot form a basis for any vector space of any dimension.