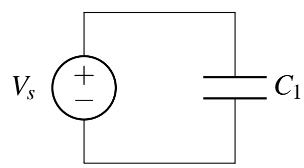
EECS 16A Designing Information Devices and Systems I
Summer 2020 Discussion 4B

1. Voltages Across Capacitors

For the circuits given below, calculate the voltage across the capacitors. For parts (a) and (b) only, also calculate the charge and energy stored in each capacitor. Let $C_1 = 1\,\mu F$, $C_2 = 3\,\mu F$, $V_s = 1\,V$, and $I_s = 2\,mA$.

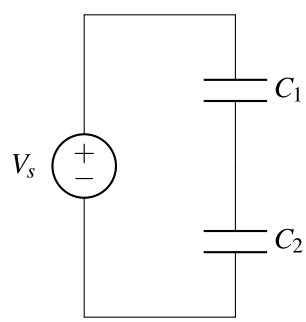
(a)



Answer:

The capacitor is charged by the voltage source, the value of which is V_s. Hence, the voltage across the capacitor has to be V_s. The charge is $q = C_1 V_s = 1\mu C$ ($+q$ accumulates on the top plate, $-q$ on the bottom plate). The energy stored is $E = \frac{C_1 V_s^2}{2} = \frac{1}{2}\mu J$.

(b)



Answer:

$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

The series equivalent capacitance is $C_{eq} = \frac{3}{4}\mu F$, so $Q_{eq} = C_{eq} V = \frac{3}{4}\mu C$. Note that this means $+\frac{3}{4}\mu C$ is on the top plate of C_1 and $-\frac{3}{4}\mu C$ is on the bottom plate of C_2. Hence, $-\frac{3}{4}\mu C$ is stored on the bottom plate of C_1 and $+\frac{3}{4}\mu C$ on the top plate of C_2. This result agrees with the conservation of charge, since the net charge on the middle node node before and after connecting the voltage source remains zero. It also means that the same amount of charge, $Q_{eq} = \frac{3}{4}\mu C$ is stored on both capacitors.

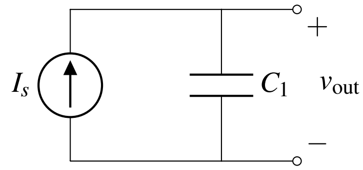
The voltage V_1 across C_1 is $\frac{Q_{eq}}{C_1} = \frac{3}{4}V$. The voltage V_2 across C_2 is $\frac{Q_{eq}}{C_2} = \frac{1}{4}V$.

The charge stored on both is Q_{eq} as mentioned above. The energy stored can be calculated as $\frac{1}{2}CV^2$ for each capacitor, so $E_1 = 280\,nJ$ and $E_2 = 94\,nJ$.

2. Current Sources And Capacitors

For the circuits given below, give an expression for $v_{\text{out}}(t)$ in terms of I_s, C_1, C_2, and t. Assume that all capacitors are initially uncharged, i.e. the initial voltage across each capacitor is 0 V.

(a)



Answer:

$$I_s = C_1 \frac{dv_{\text{out}}}{dt}(t)$$

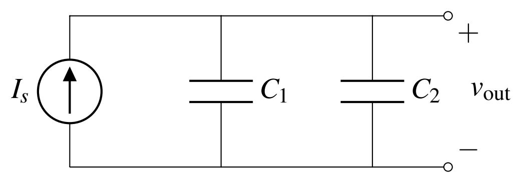
$$\int_{t=0}^{t=t} \frac{I_s}{C_1} dt = \int_{v_{\text{out}}=v_{\text{out}}(0)}^{v_{\text{out}}=v_{\text{out}}(t)} dv_{\text{out}}$$

$$\frac{I_s}{C_1}(t-0) = v_{\text{out}}(t) - v_{\text{out}}(0)$$

$$v_{\text{out}}(t) = \frac{I_s t}{C_1} + v_{\text{out}}(0)$$

Since the capacitor is initially uncharged, $v_{\text{out}}(0) = 0$, so $v_{\text{out}}(t) = \frac{I_s t}{C_1}$.

(b)



Answer:

We can combine the two capacitors into an equivalent capacitor with capacitance $C_1 + C_2$. Again, $v_{\text{out}}(0) = 0$ because all capacitors are initially uncharged.

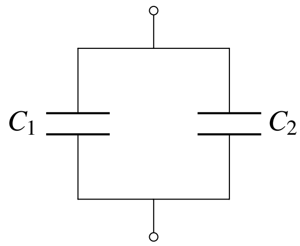
$$I_s = (C_1 + C_2)\frac{dv_{\text{out}}(t)}{dt}$$

$$v_{\text{out}}(t) = \frac{I_s t}{C_1 + C_2} + v_{\text{out}}(0) = \frac{I_s t}{C_1 + C_2}$$

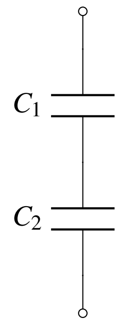
3. Practice: Series And Parallel Capacitors

Derive C_{eq} for the following circuits.

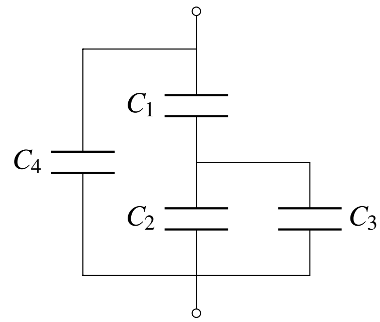
(a)



(b)



(c)

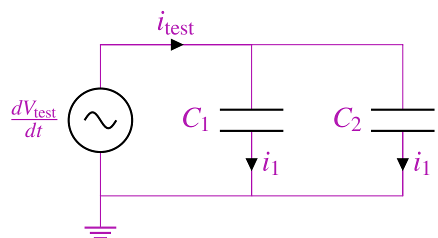


Answer:

(a)

$$C_{eq} = C_1 + C_2$$

Notice these capacitors are in parallel. We can derive their equivalent capacitance by connecting them to a voltage source with a constant derivative, as shown by the circuit below:



Since both capacitors have the same voltage across them:

$$\frac{dV_{C_1}}{dt} = \frac{dV_{C_2}}{dt} = \frac{dV_{\text{test}}}{dt}$$

$$i_1 = C_1 \frac{dV_{\text{test}}}{dt}$$

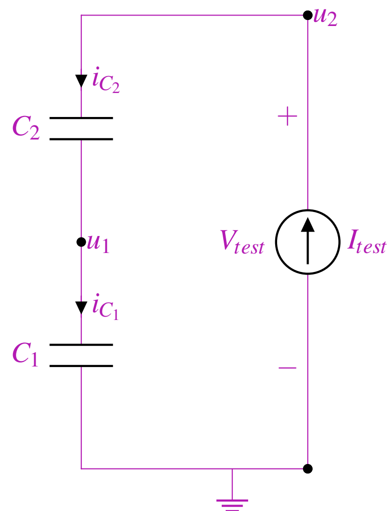
$$i_2 = C_2 \frac{dV_{\text{test}}}{dt}$$

$$i_t = i_1 + i_2 = (C_1 + C_2) \frac{dV_{\text{test}}}{dt}$$

Since we know $i_{\text{test}} = C_{\text{eq}} \frac{dV_{\text{out}}}{dt}$,

$$C_{\text{eq}} = C_1 + C_2$$

(b) In order to find the equivalence capacitance of the circuit, we plug in a test current source, and measure the rate of change of voltage across it.



From KCL, we know that all of the currents are equal.

$$i_{C_1} = i_{C_2} = I_{test}$$

For each capacitor, we plug in our $I - \frac{dV}{dt}$ relationship:

$$i_{C_1} = I_{test} = C_1 \frac{du_1}{dt}$$

$$i_{C_2} = I_{test} = C_2 \frac{d(u_2 - u_1)}{dt} = C_2 \left(\frac{du_2}{dt} - \frac{du_1}{dt} \right)$$

Next, we eliminate u_1 from the equations above and rearrange.

$$\frac{du_1}{dt} = \frac{I_{test}}{C_1} \Rightarrow I_{test} = C_2 \frac{du_2}{dt} - \frac{C_2}{C_1} I_{test}$$

$$I_{test} = \frac{C_2}{1 + \frac{C_2}{C_1}} \frac{du_2}{dt}$$

Finally, we plug in that $u_2 = V_{test}$ and solve for the equivalent capacitance with $C_{eq} = I_{test} / \frac{dV_{test}}{dt}$

$$I_{test} = \frac{C_2}{1 + \frac{C_2}{C_1}} \frac{dV_{test}}{dt}$$

$$\Rightarrow C_{eq} = \frac{C_2}{1 + \frac{C_2}{C_1}} = \frac{C_1 C_2}{C_1 + C_2}$$

Note that this is the same as saying $C_{eq} = C_1 \parallel C_2$. Remember that the $\parallel$ operator is mathematical notation; in this case, the capacitors are actually in series, but *mathematically* their equivalent circuit is found via the "parallel resistor" operation.

(c) Given that we know what the relationship for capacitors in series and parallel are from the last two parts, we can just simply the capacitors step by step:

$$C_{eq} = (C_4 + (C_1 \parallel (C_2 + C_3))) = \frac{C_4(C_1 + C_2 + C_3) + C_1(C_2 + C_3)}{C_1 + C_2 + C_3}$$